Geometric Properties of Gabor Frames and Their Applications to the Phase Retrieval Problem

by

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Abstract

In this thesis we address questions arising in two different research areas of mathematics, namely, in the area of Gabor frames in finite dimensions, belonging to the field of applied harmonic analysis, and in phase retrieval, belonging to the field of signal processing. Our work is inspired by the phase retrieval problem, which is motivated by exciting “real world” applications, such as optics, speech recognition, astronomical imaging, quantum mechanics, and wireless communication. At the same time, the study of this problem leads to beautiful and insightful mathematics at the intersection of different fields.

Frame theory and signal processing are closely linked with each other. Indeed, frames proved to be a powerful tool in many areas of applied mathematics, computer science, and engineering. In particular, frames provide a redundant, stable way of representing a signal in signal processing. The investigation of various geometric properties of frames plays a crucial role in different signal processing problems, including compressive sensing, quantization, and phase retrieval. Such properties are sufficiently well-studied for randomly generated Gaussian frames with independent frame vectors. At the same time, the concrete application for which a signal processing problem is studied usually dictates the structure of the frame used to represent a signal. This motivates the study of properties of structured, application relevant frames, such as Gabor frames.

The focus of this thesis is the investigation of geometric properties of Gabor frames and their role in the phase retrieval problem with time-frequency structured measurement frames. Phase retrieval is the non-convex inverse problem of signal reconstruction from intensity measurements with respect to a measurement frame. Even though it has been studied for a long time, until recently very little was known about how to achieve stable and efficient reconstruction, and the existing reconstruction methods lacked rigorous mathematical understanding. Nowadays, the case when the measurement frame is a Gaussian frame with independent frame vectors is sufficiently well studied. At the same time very little is known about the case of structured, application relevant frames. The main reason for this is that some geometric properties of structured frames are not yet fully understood.

In our work, we investigate such frame properties as optimal frame bounds and frame order statistics for Gabor frames with random windows. Roughly speaking, frame order statistics reflect how “flat” the vector of frame coefficients can be. The obtained results allow us to conclude that the properties of Gabor frames with random windows are often quite similar to the properties of Gaussian frames with independent vectors. This implies that Gabor frames can be used as “substitutes” for Gaussian frames, and recovery and robustness guarantees of many phase retrieval methods can be potentially adapted for time-frequency structured measurement frames. The obtained estimates on frame order statistics for Gabor frames also
imply stability of phaseless reconstruction, irrespective of the phase retrieval method used.

We design an efficient phase retrieval algorithm from a nearly optimal number of time-frequency structured measurements and show its robustness in the case when measurements are corrupted by additive noise. Robustness analysis of the constructed algorithm turns out to be also closely linked to the geometric properties of Gabor frames, and, in particular, to the frame order statistics and optimal frame bounds.
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Chapter 1

Introduction

In signal processing, we view each vector \( x \in \mathbb{C}^M \) as a signal evolving in time. That is, we interpret the coordinates \( x(m), m \in \mathbb{Z}_M \), of the vector \( x \) as the values of the analyzed signal sampled in time. With this interpretation, we have complete information about the behavior of the signal in time, but all the frequency information is hidden. For this reason, the Fourier transform became one of the major tools in analysis and signal processing. But, while the Fourier transform of a signal has perfect frequency resolution, all the information on how does the signal change in time is hidden in the phases of the Fourier coefficients. Thus, an alternative to the Fourier transform that provides information both on time and frequency behavior of a signal was required. This question was resolved by Gabor, who came up with a completely new approach to analysis and processing of signals [34]. Namely, he introduced a way to obtain a redundant signal representation that contains both time and frequency information about the signal.

Later this approach was made more rigorous, and the more general notion of a frame has been introduced, see [26, 25, 24]. More precisely, in the finite dimensional setup considered in this manuscript, we call a set of vectors \( \Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M \) a frame with frame bounds \( 0 < A \leq B \) if, for any \( x \in \mathbb{C}^M \), the following inequality holds

\[
A \|x\|_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B \|x\|_2^2.
\] (1.1)

The values \( \langle x, \varphi_j \rangle, j \in \{1, \ldots, N\} \), are called in this case the frame coefficients of \( x \) with respect to the frame \( \Phi \). We note that (1.1) holds for some \( 0 < A \leq B \) if and only if \( \text{span}(\Phi) = \mathbb{C}^M \). That is, the notion of a frame is equivalent to the notion of a spanning set of \( \mathbb{C}^M \) in the finite dimensional case. In particular, we have \( |\Phi| = N \geq M \). We identify a frame \( \Phi = \{\varphi_j\}_{j=1}^N \) with its synthesis matrix which has vectors \( \varphi_j \) as columns. To reconstruct a vector from its frame coefficients, one can use a dual frame \( \tilde{\Phi} = \{\tilde{\varphi}_j\}_{j=1}^N \), defined so that \( x = \sum_{j=1}^N \langle x, \varphi_j \rangle \tilde{\varphi}_j \), for each \( x \in \mathbb{C}^M \).

Due to overcompleteness of the set of frame vectors, a dual frame is not uniquely defined in general. The standard dual frame of \( \Phi \) is given by the Moore-Penrose pseudoinverse \( (\Phi \Phi^*)^{-1} \) of the synthesis matrix, where \( \Phi^* \) denotes the adjoint of the synthesis matrix \( \Phi \) and is called the analysis matrix of the frame \( \Phi \).

Nowadays, frames have established themselves as a standard tool in pure and applied mathematics, as well as computer science and engineering. The representation of a vector using its frame coefficients with respect to a frame \( \Phi \) is redundant, meaning that we have overcomplete information about the signal. This redundancy
is often beneficial for practical signal processing problems where frame representation is used. One of the reasons for that is flexibility of such an approach. Indeed, depending on the requirements of a concrete application, we can use a specific frame to create a more sparse representation of a signal in comparison to the standard basis representation, or, on the contrary, more “flat”, with all frame coefficients having approximately the same magnitude, see for example [54]. Moreover, provided we have a control on frame bounds, redundant frame representations of a signal are robust to additive noise, quantization, and erasures. The latter means that some of the frame coefficients are missing or cannot be used for reconstruction. Another valuable advantage of redundancy is that a frame can be designed so that the frame representation of a signal is able to capture some specific signal characteristics. This shows the importance of frames for signal processing.

In this manuscript, we consider the phase retrieval problem, where redundancy of a frame representation of a signal plays a crucial role. This problem arises naturally in many applications within a variety of fields in science and engineering, where the only available information about a signal of interest is the set of magnitudes of its frame coefficients with respect to a measurement frame. Among such applications are optics, astronomical imaging, quantum mechanics, and speech recognition.

Phase retrieval problem can be formulated as follows. Let \( \Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{C}^M \) be a frame. We consider the measurement map \( A_\Phi : \mathbb{C}^M \rightarrow \mathbb{R}^N \) defined by \( A_\Phi(x) = \{ |\langle x, \varphi_j \rangle|^2 \}_{j=1}^N \). For a given vector of measurements \( b \in \mathbb{R}^N \), we address the following non-convex inverse problem

\[
\begin{align*}
\text{find} & \quad x \\
\text{subject to} & \quad A_\Phi(x) = b.
\end{align*}
\]

Since \( A_\Phi(x) = A_\Phi(e^{i\theta}x) \) for any \( \theta \in [0, 2\pi) \), the initial signal \( x \) can be reconstructed in the best case only up to a global phase factor. Thus, we identify each \( x \in \mathbb{C}^M \) with its up-to-a-global-phase equivalence class and consider the measurement map \( A_\Phi \) to be defined on the set of equivalence classes \( \mathbb{C}^M/\sim \).

We note that phase retrieval is a non-convex inverse problem in its core. Thus, even assuming injectivity of the measurement map \( A_\Phi \), it is NP-hard in general [71]. Therefore, besides investigation of injectivity and stability of the measurement map, an important task in phase retrieval is to construct efficient recovery algorithms for specific choices of the measurement frame \( \Phi \).

The structure of the measurement frame \( \Phi \) is usually dictated by a concrete application where the phase retrieval problem arises. For instance, measurements arising in optics and diffraction imaging are of the form of pointwise squared absolute values of masked Fourier transforms of the object. That is, the measurement map is given by

\[
A_{\mathcal{F}, \{ f_j \}_{j \in I}}(x) = \{ |\mathcal{F}(x \odot f_j)(\ell)|^2 \}_{j \in I, \ell \in \mathbb{Z}^M},
\]

where \( \mathcal{F} \) denotes the Fourier transform, \( \{ f_j \}_{j \in I} \subset \mathbb{C}^M \) is the set of known masks, and \( \odot \) denotes the pointwise multiplication. In some other applications, including speech recognition and radars, the arising measurement frames are Gabor frames, see Definition 2.1.1.

At the same time, recovery guarantees for most of the existing phase retrieval algorithms, such as PhaseLift [13] and Wirtinger flow algorithm [15], are proven only for randomly generated frames with independent vectors, such as Gaussian frames. However, measurements with respect to such frames are not implementable in practical applications.
1.1 Overview of main results and thesis structure

The thesis is organized in the following way. The remaining part of this chapter is devoted to an overview of the main results obtained in this thesis.

In Chapter 2 we provide a brief overview of the basic notions and results from finite frame theory and time-frequency analysis, probability theory, and expander graphs theory, which we use throughout the manuscript.

Chapter 3 discusses geometric properties of Gabor frames and compares them to the corresponding properties of random frames with independent frame vectors. We start with an overview of the previously obtained results on spark, coherence, and restricted isometry property (Section 3.1). Then we turn to the investigation of frame order statistics as introduced in Section 3.2, and also optimal frame bounds of Gabor frames and their robustness to erasures, which we study in Section 3.3. Frame order statistics for random frames with independent vectors are studied in Appendix A.

The phase retrieval problem in the case that the measurement frame has time-frequency structure is considered in Chapter 4. There, in Section 4.1 we describe some applications where this problem arises and then provide an overview of the state of art results on phase retrieval in Section 4.2. The problem of stability of the phaseless measurement map for random frames with independent vectors of bounded fourth moment and Gabor frames with a random window is discussed in Section 4.3. We also construct an efficient reconstruction algorithm from phaseless time-frequency structured measurements in Section 4.5 and prove robustness guarantees for it in Section 4.5.2.

We now give an overview of the main results obtained in each chapter in more details.

**Geometric properties of random frames**

In Chapter 3 that is dedicated to geometric properties of Gabor frames with a random window, the two main objectives of our study are frame order statistics and
CHAPTER 1. INTRODUCTION

distribution of the singular values of the analysis matrix of a frame.

In Section 3.2, we study the “flatness” of frame coefficients. This property of frames is of major importance for phase retrieval problem, as discussed in Section 4.5, and also for quantization [54, 20] and other problems in signal processing and frame theory. Roughly speaking, for a given frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$, we aim to find constants $K > c > 0$, such that, for every $x \in \mathbb{S}^{M-1}$, most of the frame coefficients $\langle x, \varphi_j \rangle$, $\varphi_j \in \Phi$, are in the range $\frac{c}{\sqrt{M}} \leq |\langle x, \varphi_j \rangle| \leq \frac{K}{\sqrt{M}}$. We formalize this in the following definition.

**Definition 1.1.1.** Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{C}^M$.

(i) For $\alpha \leq N$, the $\alpha$-smallest frame order statistics of $\Phi$ is given by

$$S_{\text{FOS}}(\Phi, \alpha, x) = \max_{J \subseteq \{1, \ldots, N\}, |J| \geq \alpha} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

We define the $\alpha$-smallest uniform frame order statistics of $\Phi$ as

$$S_{\text{uFOS}}(\Phi, \alpha) = \min_{x \in \mathbb{S}^{M-1}} S_{\text{FOS}}(\Phi, \alpha, x).$$

(ii) For $\beta \leq N$, the $\beta$-largest frame order statistics of $\Phi$ is given by

$$L_{\text{FOS}}(\Phi, \beta, x) = \min_{J \subseteq \{1, \ldots, N\}, |J| \geq \beta} \max_{j \in J} |\langle x, \varphi_j \rangle|.$$

We define the $\beta$-largest uniform frame order statistics of $\Phi$ as

$$L_{\text{uFOS}}(\Phi, \beta) = \max_{x \in \mathbb{S}^{M-1}} L_{\text{FOS}}(\Phi, \beta, x).$$

As follows from the definition, deleting $N - \alpha$ smallest and $N - \beta$ largest in modulus frame coefficients ensures that the remaining coefficients satisfy $S_{\text{FOS}}(\Phi, \alpha) \leq |\langle x, \varphi_j \rangle| \leq L_{\text{FOS}}(\Phi, \beta)$, for every vector $x \in \mathbb{S}^{M-1}$.

We obtain the following result on the distribution of frame coefficients for the case when the frame vectors are independent random variables in Appendix A.

**Theorem 1.1.2.** Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ (with $M$ large enough) be a frame, such that $\varphi_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, are independent identically distributed centered random variables normalized so that $\text{Var}(\varphi_j(m)) = \frac{1}{M}$. Assume further that $\mathbb{E}(|\varphi_j(m)|^4) \leq \frac{B}{M^2}$ and $N \geq C_0 M \log M$ for some constants $B \geq 1$ and $C_0 > 0$ not depending on $M$. Then the following holds.

(a) For each $\alpha < 1 - \frac{1}{2C_0}$, there exist constants $c, c_1 > 0$ depending only on $B$, $\alpha$, $C_0$, such that

$$S_{\text{uFOS}}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}}$$

with probability at least $1 - e^{-c_1 M \log M}$.

(b) For each $\beta < 1 - \frac{1}{2C_0}$, there exist constants $K, c_2 > 0$ depending only on $B$, $\beta$, $C_0$, such that

$$L_{\text{uFOS}}(\Phi, \beta N) \leq \frac{K}{\sqrt{M}}$$

with probability at least $1 - e^{-c_2 M \log M}$.
1.1. OVERVIEW OF MAIN RESULTS AND THESIS STRUCTURE

Theorem 1.1.2 generalizes [1, Lemmas 6.9 and 6.10], where similar bounds are shown for Gaussian frames. However, our result provides bounds for much larger class of random frames, which include, in particular, all subgaussian random frames.

The case of Gabor frames $\Phi_\Lambda = (g, \Lambda)$ with a randomly generated window $g$ is considered in Section 3.2. In particular, we prove the following result.

**Theorem 1.1.3.** Fix $x \in \mathbb{S}^{M-1}$ and consider a Gabor frame $\Phi_\Lambda = (g, \Lambda)$ with $\Lambda \subseteq \mathbb{Z}_M \times \mathbb{Z}_M$ and a random window $g$ uniformly distributed on the unit sphere $\mathbb{S}^{M-1}$. Then the following holds.

(a) For any $c \in (0, 1)$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have

$$S_{FOS}(\Phi_\Lambda, |\Lambda|(c^2 + kc), x) \geq \frac{c}{\sqrt{M}}.$$

(b) For any $C > 1$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have

$$L_{FOS}(\Phi_\Lambda, |\Lambda|\left(\frac{8}{\pi}e^{-c^2} + k \frac{2\sqrt{\pi}}{C}e^{-C^2}\right), x) \leq \frac{C}{\sqrt{M}}.$$

Even though Theorem 1.1.3 is formulated for Gabor frames, the analogous result holds for a much more general class of frames with frame vectors uniformly distributed on $\mathbb{S}^{M-1}$.

Theorem 1.1.3 is a non-uniform analog of Theorem 1.1.2 in the setting of Gabor frames, meaning that the proven bounds hold with high probability for each individual $x$. Theorem 1.1.3 is a crucial ingredient for the proof of the non-uniform robustness guarantees for the phase retrieval algorithm constructed in Chapter 4.

We obtain the following uniform bound on the number of large frame coefficients in the Gabor setting in Section 3.2.

**Theorem 1.1.4.** Consider a Gabor frame $\Phi_\Lambda = (g, \Lambda)$ with a random Gaussian window $g$, such that $g(m) \sim \text{i.i.d. } \mathcal{CN}(0, \frac{1}{\sqrt{M}})$. Then, for some suitably chosen numerical constants $c, c_1 > 0$,

$$L_{uFOS}(\Phi_\Lambda, cM \log^4 M) \leq \sqrt{\frac{3}{2c} \log^2 M},$$

with probability at least $1 - e^{-c_1 \log^3 M}$.

Establishing a uniform version of Theorem 1.1.3 is a crucial task for further investigation of the geometric properties of Gabor frames. Among other things, it would lead to a large step forward in phase retrieval with Gabor frames, as we discuss in Section 4.3.2.

Numerical results that provide an evidence of nice “flatness” properties of Gabor frames with a random window are presented and analyzed in Section 3.2.1.

In Section 3.3, we study the optimal frame bounds of Gabor frames with a random Steinhaus window. This property is of major importance in many signal processing problems, since frame bounds which are sufficiently close to each other imply robust reconstruction of a signal from its frame coefficients.

In particular, we obtain the following result for the upper frame bound.
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Theorem 1.1.5. Let $g$ be a Steinhaus window, that is, $g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}$, $m \in \mathbb{Z}_M$, with $y_m$ independent uniformly distributed on $[0, 1)$. Consider a Gabor system $(g, \Lambda)$ with $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$. Then, for each fixed $\varepsilon \in (0, 1)$, with probability at least $1 - \varepsilon$, 

$$\sigma^2_{\max}(\Phi_{\Lambda}^*) \leq |\Lambda| M + \sqrt{\frac{|\Lambda|}{\varepsilon} \left(1 - \frac{|\Lambda|}{M^2}\right)}.$$

For the case when we choose the set $\Lambda$ at random, we show the following result on both upper and lower frame bounds.

Theorem 1.1.6. Let $g$ be a Steinhaus window. For any fixed even $m \in \mathbb{N}$, consider a Gabor system $(g, \Lambda)$ with a random set $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ constructed so that events $\{(k, \ell) \in \Lambda\}$ are independent for all $(k, \ell) \in \mathbb{Z}_M \times \mathbb{Z}_M$ and have probability $\tau = \frac{C \log M}{M^m}$, for a sufficiently large constant $C > 0$ depending on $m$. Then, with high probability (with respect to the choice of $\Lambda$),

$$\mathbb{P}\left\{ \frac{|\Lambda|}{M} (1 - \delta) \leq \sigma^2_{\min}(\Phi_{\Lambda}^*) \leq \sigma^2_{\max}(\Phi_{\Lambda}^*) \leq \frac{|\Lambda|}{M} (1 + \delta) \right\} \geq 1 - \varepsilon,$$

where $\varepsilon \in (0, 1)$ depends on $m$, $\delta$, and the choice of $C$.

Numerical results described and discussed in Section 3.3.1 suggest that the obtained bounds, and, in particular Theorem 1.1.6, can be further improved. Moreover, the obtained numerical results allow us to conjecture that Gabor frames with a random window are robust to erasures, in the sense that frame bounds of all sufficiently large subframes of a Gabor frame are uniformly bounded. This property is significant for such applications as communication systems, where signal transmission can be interrupted, and also for robustness of the phase retrieval method obtained in Section 4.5, as follows from the proof of Theorem 1.1.10.

Phase retrieval with Gabor measurements

Section 4.3 is dedicated to the study of stability of the phaseless measurement map $\mathcal{A}_\Phi$ for a measurement frame $\Phi \subset \mathbb{R}^M$. The notion of the phaseless measurement map stability, irrespective of the reconstruction method used, was introduced by Eldar and Mendelson [27].

More precisely, for a measurement frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{R}^M$, we call the phaseless measurement map $\mathcal{A}_\Phi : \mathbb{R}^M \to \mathbb{R}^N$, given by $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$, stable with a constant $C$ in a set $T \subset \mathbb{R}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C||x - y||_2||x + y||_2.$$

We note that stability in a set is a much stronger property than invertibility up to a global phase.

In Section 4.3, we consider two different classes of measurement frames, namely, random frames with independent vectors of bounded fourth moment and Gabor frames with a random window. For random frames with independent frame vectors and under the additional fourth moment assumption, the following stability result is shown in Section 4.3.1.
Theorem 1.1.7. Let a frame $\Phi = \{\varphi_j\}_{j=1}^{N} \subset \mathbb{R}^M$, with $M$ large enough, be such that $\varphi_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, are independent identically distributed centered random variables, normalized so that $\text{Var}(\varphi_j(m)) = \frac{1}{M}$. Assume further that $\mathbb{E}(|\varphi_j(m)|^4) \leq \frac{B}{M^2}$, for some constant $B \geq 1$, and $N \geq C_0 M \log M$, for some constant $C_0 > 1$. Then there exists a numerical constant $L > 0$, such that, with overwhelming probability, the measurement map $A_\Phi$ is stable with constant $C \geq L \log(M)$ in $\mathbb{R}^M$. In other words, for any $x, y \in \mathbb{R}^M$,

$$||A_\Phi(x) - A_\Phi(y)||_1 \geq L \log(M)||x - y||_2||x + y||_2.$$

A non-uniform stability of the measurement map in the case of a Gabor frame with a random window is obtained in Section 4.3.2.

Theorem 1.1.8. Let $(g, \Lambda) \subset \mathbb{R}^M$ be a Gabor system with $|\Lambda| > M$ and a random window $g$ uniformly distributed on the real unit sphere $S_{2^{M-1}}^1$. Then there exists a numerical constant $C > 0$, such that for each pair $x, y \in \mathbb{R}^M$ the following holds with high probability

$$||A_\Lambda(x) - A_\Lambda(y)||_1 \geq C||x - y||_2||x + y||_2.$$

That is, $A_\Lambda$ is stable in $\mathbb{R}^M$.

In Sections 4.4 and 4.5, we design a phase retrieval algorithm, which is based on the polarization approach introduced in Section 4.2.2, see also [1, 6]. The idea of the algorithm is the following. Suppose we managed to determine the phases $\{\text{arg}(x, \varphi_j)\}_{j=1}^{N}$ of frame coefficients in addition to phaseless measurements $A_\Phi(x) = \{|x, \varphi_j|^2\}_{j=1}^{N}$. Then one can reconstruct the signal $x$ by least-squares estimation. To obtain these phases, we use some carefully selected additional phaseless measurements of $x$ with respect to polarized linear combinations of the frame vectors.

More precisely, we consider measurement frames of the form $\Phi = \Phi_\Lambda \cup \Phi_E \subset \mathbb{C}^M$, where $\Phi_\Lambda$ is a Gabor frame and $\Phi_E$ is a set of vectors needed for additional measurements. We define

1. $\Phi_\Lambda = (g, \Lambda)$, where $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ is of cardinality $|\Lambda| = O(M)$, and the random window $g \in \mathbb{C}^M$ is uniformly distributed on the complex unit sphere $S_{2^{M-1}}^1 \subset \mathbb{C}^M$.

2. $\Phi_E = \{\pi(\lambda_1)g + \omega^t \pi(\lambda_2)g, \text{ s.t. } (\lambda_1, \lambda_2) \in E, \ t \in \{0, 1, 2\}\}$, where $\omega = e^{2\pi i/3}$ and $E \subset \Lambda \times \Lambda$ is selected in a special way, so that $|E| = O(M \log M)$.

We note that measurements taken with respect to the frame $\Phi$ have form of squared modulus of masked Fourier transforms coefficients. In particular, for any $(k, \ell) \in \Lambda$, we have $|\langle x, \pi(k, \ell)g \rangle|^2 = |\mathcal{F}(x \circ T_k \bar{g})(\ell)|^2$. The measurements with respect to $\Phi_E$ have a similar form, but with a different set of masks, see Section 4.4.1 for the details.

Measurements of this type have several advantages in comparison to randomly generated frames with independent vectors. In addition to being relevant for many applications, this measurements can be efficiently implemented using fast Fourier transform, which allows a noticeable speed up of the measurement and reconstruction processes.
Based on the idea of polarization, in Section 4.5 we design the phase retrieval algorithm (Algorithm 7). In this algorithm, we construct the graph of measurements \( G = (\Lambda, E) \). The vertices of \( G \) correspond to the phaseless measurements taken with respect to the frame \( \Phi = (g, \Lambda) \), and each edge \((\lambda_1, \lambda_2) \in E\) is labeled by \( A_{\lambda_1, \lambda_2} \), which approximates the relative phase \( \omega_{\lambda_1, \lambda_2} = \left( \frac{\langle x, \pi(\lambda_1)g \rangle}{|\langle x, \pi(\lambda_1)g \rangle|} \right) - \left( \frac{\langle x, \pi(\lambda_2)g \rangle}{|\langle x, \pi(\lambda_2)g \rangle|} \right) \) between the frame coefficients \( \langle x, \pi(\lambda_1)g \rangle \) and \( \langle x, \pi(\lambda_2)g \rangle \), provided both of them are nonzero.

By polarization identity given in Lemma 4.2.15, \( \omega_{\lambda_1, \lambda_2} = A_{\lambda_1, \lambda_2} \) in the noiseless case.

Now, when we know not only the phaseless measurements with respect to \( \Phi = (g, \Lambda) \) but also the relative phases between some pairs of frame coefficients, we can reconstruct signal \( x \) using the angular synchronization method summarized in Algorithm 5, see also [74], which computes the phases of the frame coefficients. Using this algorithm, we can obtain phases only inside a connected component of the graph of measurements. Consequently, to ensure reconstruction, we need to have a sufficiently big subgraph \( G' = (\Lambda', E') \), such that \( \Phi = (g, \Lambda') \) is a frame and \( G' \) has good connectivity properties. We find such a subgraph using spectral clustering, described in Algorithm 6, see also [62].

In Section 4.4, we prove the following recovery guarantees in the noiseless case.

**Theorem 1.1.9.** Let frames \( \Phi = (g, \Lambda) \) be as considered above, with \( |\Lambda| = 12M \) and \( |E| \geq C_0 M \log M \), for \( C_0 > 0 \) sufficiently large. Then every signal \( x \in \mathbb{C}^M \) can be reconstructed exactly (up to a global phase) using Algorithm 7 with probability 1.

In the case when measurements are corrupted by additive noise, the following robustness guarantees are shown in Section 4.5.2.

**Theorem 1.1.10.** For a fixed signal \( x \in \mathbb{C}^M \), consider the measurement procedure described above with \( |\Lambda| = CM \) and \( |E| \geq C_0 M \log M \), where \( C \) and \( C_0 \) are sufficiently large. Suppose the noise vector \( \nu \) satisfies \( \frac{||\nu||_2}{||x||_2} \leq \frac{C_1}{M} \) for some \( C_1 \) small enough. Then there exists a constant \( C'' \), such that the output \( \tilde{x} \) of Algorithm 7 satisfies the following inequality with overwhelming probability.

\[
\min_{\theta \in [0, 2\pi]} ||\tilde{x} - e^{i\theta} x||_2^2 \leq \frac{C'' \sqrt{M} ||\nu||_2^2}{\Delta}
\]

with \( \Delta = \min \{ \sigma_{\min}^2(\Phi_{\Lambda'}^*) : \Lambda' \subset \Lambda, |\Lambda'| \geq \frac{2}{3} |\Lambda| \} \), where \( \Phi_{\Lambda'} \) is the synthesis matrix of the frame \( (g, \Lambda') \) and \( \sigma_{\min}(A) \) denotes the smallest singular value of a matrix \( A \).

To the best of our knowledge, this result provides the best existing robustness guarantee for phase retrieval with time-frequency structured measurements. Numerical results illustrating the behavior of the algorithm in presence of noise are given in Section 4.5.3.

We note that the stability results obtained in Theorems 1.1.7 and 1.1.8, as well as the robustness result of Theorem 1.1.10 strongly rely on the results obtained in Chapter 3.
Chapter 2

Notation and background

This chapter contains an overview of the standard notions and results in frame theory, probability theory, and expander graphs theory, that are use in the manuscript, and may be skipped by the reader. We start with introducing the notation which is used throughout this manuscript.

Here and in the sequel, ⊙ denotes pointwise multiplication of two vectors of the same dimension. We view a vector \( x \in \mathbb{C}^M \) as a function \( x : \mathbb{Z}_M \rightarrow \mathbb{C} \), that is, all the operations on indices are done modulo \( M \) and \( x(m - k) = x(M + m - k) \). We denote the complex unit sphere by \( S^{M-1} = \{ x \in \mathbb{C}^M, ||x||_2 = 1 \} \).

The adjoint matrix of \( A \in \mathbb{C}^{k \times m} \) is denoted by \( A^* \in \mathbb{C}^{m \times k} \), and the smallest and the largest singular values of \( A \) are denoted by \( \sigma_{\min}(A) \) and \( \sigma_{\max}(A) \), respectively.

The Kronecker product of two matrices \( A \) and \( B \) is denoted by \( A \otimes B \). For each \( M \in \mathbb{N} \), we denote the identity \( M \times M \) matrix by \( I_M \). The \( M^2 \)-dimensional real vector space of Hermitian matrices is denoted by \( H^M \). Also, by a slight abuse of notation, we identify a frame \( \Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{C}^M \) with its synthesis matrix, having the frame vectors \( \varphi_j \) as columns. For any \( V \subset \{ 1, \ldots, N \} \), we set \( \Phi_V = \{ \varphi_j \}_{j \in V} \).

We denote the Bernoulli distribution with success probability \( p \) by \( B(1,p) \). Further, \( N(\mu, \sigma) \) denotes the Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \), and \( \mathbb{C}N(\mu, \sigma) \) denotes the (circularly-symmetric) complex valued Gaussian distribution, where.

2.1 Frame theory and Gabor analysis in finite dimensions

In the finite dimensional setup considered in this manuscript, we call a set of vectors \( \Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{C}^M \) a frame with frame bounds \( 0 < A \leq B \) if, for any \( x \in \mathbb{C}^M \), the following inequality holds

\[
A ||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B ||x||_2^2.
\]

The values \( \langle x, \varphi_j \rangle, j \in \{ 1, \ldots, N \} \), are called in this case the frame coefficients of \( x \) with respect to the frame \( \Phi \). We note that the above inequality holds for some \( 0 < A \leq B \) if and only if \( \text{span} (\Phi) = \mathbb{C}^M \). That is, the notion of a frame is equivalent to the notion of a spanning set of \( \mathbb{C}^M \) in the finite dimensional case. In particular, we have \( |\Phi| = N \geq M \).
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A frame $\Phi = \{\varphi_j\}_{j=1}^N$ with $||\varphi_j||_2 = 1$ for all $j \in \{1, \ldots, N\}$ is called a *unit norm frame*. In the case when frame bounds can be chosen so that $A = B$, the frame $\Phi$ is called *tight*.

To reconstruct a vector from its frame coefficients, one can use a *dual frame* $\tilde{\Phi} = \{\tilde{\varphi}_j\}_{j=1}^N$, defined so that $x = \sum_{j=1}^N \langle x, \varphi_j \rangle \tilde{\varphi}_j$, for each $x \in \mathbb{Z}_M$. A dual frame is not uniquely defined if $|\Phi| > M$. The *standard dual frame* of $\Phi$ is given by the Moore-Penrose pseudoinverse $(\Phi^* \Phi)^{-1} \Phi$ of the synthesis matrix $\Phi$, where the adjoint $\Phi^*$ of the synthesis matrix is called the *analysis matrix* of the frame $\Phi$. The product $\Phi \Phi^*$ is called the *frame operator* of the frame $\Phi$.

For a complete background on frames in finite dimensions, we refer the reader to [18]. We now restrict our consideration to the special class of Gabor frames.

Let us first define two families of unitary operators on $\mathbb{C}^M$, namely, cyclic shift operators and modulation operators, and Gabor systems.

**Definition 2.1.1.**

1. *Translation* (or *time shift*) by $k \in \mathbb{Z}_M$, is given by
   
   $T_k x = T_k (x(0), x(1), \ldots, x(M-1)) = (x(m-k))_{m \in \mathbb{Z}_M}$.
   
   That is, $T_k$ permutes entries of $x$ using $k$ cyclic shifts.

2. *Modulation* (or *frequency shift*) by $\ell \in \mathbb{Z}_M$ is given by
   
   $M_\ell x = M_\ell (x(0), x(1), \ldots, x(M-1)) = \left( e^{2\pi i \ell m/M} x(m) \right)_{m \in \mathbb{Z}_M}$.
   
   That is, $M_\ell$ multiplies $x = x(\cdot)$ pointwise with the harmonic $e^{2\pi i \ell (\cdot)/M}$.

3. The superposition $\pi(k, \ell) = M_\ell T_k$ of translation by $k$ and modulation by $\ell$ is a *time-frequency shift operator*.

4. For $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the set of vectors
   
   $(g, \Lambda) = \{\pi(k, \ell)g\}_{(k,\ell) \in \Lambda}$
   
   is called the *Gabor system* generated by the *window* $g$ and the set $\Lambda$. A Gabor system which spans $\mathbb{C}^M$ is a frame and is referred to as a *Gabor frame*.

The *discrete Fourier transform* $\mathcal{F} : \mathbb{C}^M \rightarrow \mathbb{C}^M$ plays a fundamental role in Gabor analysis. It is given pointwise by

$$\mathcal{F} x(\ell) = \sum_{m \in \mathbb{Z}_M} x(m) e^{-2\pi im\ell/M}, \quad \ell \in \mathbb{Z}_M.$$  

The *short-time Fourier transform* (or *windowed Fourier transform*) $V_g : \mathbb{C}^M \rightarrow \mathbb{C}^{M \times M}$ with respect to the window $g \in \mathbb{C}^M \setminus \{0\}$ is given by

$$V_g x(k, \ell) = \langle x, \pi(k, \ell)g \rangle = \mathcal{F}(x \odot T_k \bar{g})(\ell), \quad k, \ell \in \mathbb{Z}_M.$$  \hspace{1cm} (2.1)

Equality (2.1) indicates that the short-time Fourier transform on $\mathbb{C}^M$ can be efficiently computed using the *fast Fourier transform* (FFT), an efficient algorithm to compute the discrete Fourier transform of a vector. Phase retrieval with time-frequency structured measurements benefits from this, as it reduces the run time of a recovery algorithm.

We postpone the discussion of the properties of Gabor frames till Chapter 3. A more detailed description of Gabor frames in finite dimensions and their properties can be found in [68].
2.2 Probability theory tools

In this manuscript, we consider Gabor frames with random windows, for several different probability distributions, including Steinhaus, Gaussian, $L$-subgaussian, and uniform on $\mathbb{S}^{M-1}$ distributions. In this section we summarize their main properties, which we rely on in subsequent chapters.

Let $X$ be a complex-valued random variable. We say that $X$ is centered if it has zero expectation, that is, $\mathbb{E}(X) = 0$. For $n \in \mathbb{N}$, the $n$th moment of $X$ is given by $\mathbb{E}(|X - \mathbb{E}(X)|^n)$. A random vector $g \in \mathbb{C}^M$ is called isotropic if for every $h \in \mathbb{C}^M$,

$$\mathbb{E}(|\langle g, h \rangle|^2) = ||h||^2_2.$$ 

One of the main probability distributions considered in this work is Steinhaus distribution defined in the following way.

**Definition 2.2.1.** A random vector $g \in \mathbb{S}^{M-1}$, such that $g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}$, $m \in \mathbb{Z}_M$ with $y_m$ independent uniformly distributed on $[0, 1)$, is called a Steinhaus vector.

Another important and quite general class of random variables is $L$-subgaussian random variables. The tail probability of a subgaussian random variable is dominated by the tail probability of a Gaussian random variable. More precisely, we have the following definition.

**Definition 2.2.2.** A centered random variable $X$ is called subgaussian if there exists $L > 0$, such that

$$P\{|X| > t\} \leq 2 e^{-\frac{t^2}{L^2}},$$

for all $t > 0$. Equivalently, $X$ is called subgaussian if there exists $L > 0$, such that, for any $n \in \mathbb{N}$, $X$ satisfies the following moment assumption

$$(\mathbb{E}(|X|^n))^{\frac{1}{n}} \leq CL\sqrt{n},$$

where $C$ is a constant not depending on $n$. The minimal $L$ for which this inequality holds is called the subgaussian moment of $X$. If $L$ is the subgaussian moment of $X$, then $X$ is called $L$-subgaussian.

The class of subgaussian random variables is quite large and includes, in particular, all bounded random variables.

For completeness, we also include in this section some classical inequalities estimating tale probabilities of random variables that we use in the sequel.

**Lemma 2.2.3 (Markov’s inequality).** For any random variable $X$, such that $X \geq 0$ with probability one, and every $t > 0$,

$$P\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}.$$ 

A well-known consequence of Markov’s inequality is the following Chebyshev’s inequality, which provides a better tail probability estimate.

**Lemma 2.2.4 (Chebyshev’s inequality).** For any random variable $X$ and $t > 0$,

$$P\{|X - \mathbb{E}(X)| \geq t\} \leq \frac{\text{Var}(X)}{t^2}.$$
We also state here Hoeffding’s inequality in the special case of Bernoulli random variables.

**Lemma 2.2.5** (Hoeffding’s inequality). Let $X_j$, $j \in \{1, \ldots, N\}$, be independent identically distributed Bernoulli random variables, such that $\mathbb{P}\{X_j = 1\} = p$, for some $p \in (0, 1)$, that is $X_j \sim \text{i.i.d. } B(1, p)$. Consider the random variable $S = \sum_{j=1}^{N} X_j$. Then, for every $t > 0$, we have
\[
\mathbb{P}\{S < (p - t)N\} \leq e^{-2t^2N} \quad \text{and} \quad \mathbb{P}\{S > (p + t)N\} \leq e^{-2t^2N}.
\]

The following lemma, proven in [50], is useful for obtaining bounds on the norms of random vectors.

**Lemma 2.2.6.** [50] Let $Y_1, \ldots, Y_M \sim \text{i.i.d. } \mathcal{N}(0, 1)$ and fix $c = (c_1, \ldots, c_M)$ with $c_k \geq 0$, $k \in \{1, \ldots, M\}$. Then, for $Z = \sum_{k=1}^{M} c_k(Y_k^2 - 1)$ the following inequalities hold for any $t > 0$.
\[
\begin{align*}
\mathbb{P}\{Z \geq 2||c||_2 \sqrt{t} + 2||c||_\infty t\} &\leq e^{-t}; \\
\mathbb{P}\{Z \leq -2||c||_2 \sqrt{t}\} &\leq e^{-t}.
\end{align*}
\]

Using Lemma 2.2.6, we obtain the following bounds on the norm of a random Gaussian vector $h \sim \mathcal{CN}(0, \frac{1}{M} I_M)$.

**Lemma 2.2.7.** Consider a random vector $h \in \mathbb{C}^M$, such that $h \sim \mathcal{CN}(0, \frac{1}{M} I_M)$. Then, there exists a constant $C > 0$, such that
\[
\mathbb{P}\left\{ \frac{1}{2} < ||h||_2 < 2 \right\} \geq 1 - e^{-CM}.
\]

**Proof.** First, we note that
\[
2M||h||_2^2 = 2M \sum_{k=1}^{M} (|a_k|^2 + |b_k|^2),
\]
where $h(k) = a(k) + ib(k)$ and $a(k), b(k) \sim \text{i.i.d. } \mathcal{N}(0, \frac{1}{2M})$. Then, for any $k \in \{1, \ldots, M\}$, $\sqrt{2M}a(k), \sqrt{2M}b(k)$ are independent standard Gaussian random variables. We apply inequality (2.2) from Lemma 2.2.6 with $c_k = 1$, $k \in \{1, \ldots, M\}$, to obtain that, for any $t > 0$,
\[
\mathbb{P}\{2M||h||_2^2 \geq \sqrt{8Mt} + 2t + 2M\} \leq e^{-t}.
\]
Taking $t = M/2$, we have
\[
\mathbb{P}\{||h||_2^2 > 4\} = \mathbb{P}\{2M||h||_2^2 > 8M\} \leq \mathbb{P}\{2M||h||_2^2 \geq 5M\} \leq e^{-M/2}.
\]
Similarly, by applying inequality (2.3) from Lemma 2.2.6 with $c_k = 1$, we get
\[
\mathbb{P}\left\{ ||h||_2^2 \leq -\sqrt{\frac{2t}{M}} + 1 \right\} \leq e^{-t},
\]
for every $t > 0$. Taking $t = 9M/32$, we obtain
\[
\mathbb{P}\left\{ ||h||_2^2 \leq -\sqrt{\frac{2t}{M}} + 1 \right\} = \mathbb{P}\left\{ ||h||_2^2 \leq \frac{1}{4} \right\} \leq e^{-9M/32},
\]
Summarizing the bounds obtained in (2.4) and (2.5), we conclude the desired claim. \qed
2.2.1 Fourier bias

In additive combinatorics, the notion of Fourier bias is used to measure pseudorandomness of a set. Roughly speaking, it helps to distinguish between sets which are highly uniform and behave like random sets, and those which are highly non-uniform and behave like arithmetic progressions [76].

Definition 2.2.8. Take \( C \subseteq \mathbb{Z}_M \) and let \( 1_C \) be the characteristic function of \( C \). Then the Fourier bias of \( C \) is given by

\[
||C||_u = \max_{m \in \mathbb{Z}_M \setminus \{0\}} |( \mathcal{F}1_C )(m) |.
\]

As discussed in [76], the Fourier bias satisfies the following properties.

1. For every \( C \subseteq \mathbb{Z}_M \), \( ||C||_u \geq 0 \). Moreover, \( ||C||_u = 0 \) if and only if \( C = \mathbb{Z}_M \) or \( C = \emptyset \).

2. The Fourier bias is invariant under the symmetries. More precisely, for a set \( C \subseteq \mathbb{Z}_M \), we have \( ||C||_u = || -C||_u = ||C - m||_u = ||\mathbb{Z}_M \setminus C||_u \), where \( -C = \{ -k, k \in C \} \) and \( C - m = \{ k - m, k \in C \}, \) for \( m \in \mathbb{Z}_M \).

3. Inclusion \( A \subseteq B \subseteq \mathbb{Z}_M \) does not imply \( ||A||_u \leq ||B||_u \).

4. A triangular inequality holds for the Fourier bias. That is, for \( A, B \subseteq \mathbb{Z}_M \), such that \( A \cap B = \emptyset \), we have \( ||A||_u - ||B||_u \leq ||A \cup B||_u \leq ||A||_u + ||B||_u \).

The following lemma follows from Chernoff’s inequality and can be found in [76, Lemma 4.16]. Loosely speaking, it shows that, if \( B \) is a random subset of \( A \subseteq \mathbb{Z}_M \), then \( ||B||_u \) is tightly concentrated around \( \frac{|B|}{|A|}||A||_u \). In other words, the Fourier bias of a random subset scales proportionally to its cardinality.

Lemma 2.2.9. Consider an additive subset \( A \) of \( \mathbb{Z}_M \) with \( M > 4 \), and fix \( 0 < \tau \leq 1 \). Let \( B \) be a random subset of \( A \), such that \( 1_B(a) \sim \text{i.i.d.} \) \( B(1, \tau) \), for \( a \in A \), that is, events \( \{ a \in B \} \) are independent and have probability \( \tau \). Then, for any \( \lambda > 0 \) and \( \sigma^2 = \frac{|A|}{M^2 \tau} (1 - \tau) \), we have

\[
\mathbb{P} \{ ||B||_u - \tau ||A||_u \geq \lambda \sigma \} \leq 4M \max \left\{ e^{-\frac{\lambda^2}{8}}, e^{-\frac{\lambda\sigma}{2\sqrt{2}}} \right\}.
\]

As an easy consequence of Lemma 2.2.9, we obtain the following result that provides an efficient bound on the absolute value of the sum of randomly sampled roots of unity.

Corollary 2.2.10. Let \( B \) be a random subset of \( \mathbb{Z}_M \), such that \( 1_B(m) \sim \text{i.i.d.} \) \( B(1, \tau) \), for \( m \in \mathbb{Z}_M \) and \( 0 < \tau < 1 \). Then, for any constant \( C > 4\sqrt{2} \), we have

\[
\mathbb{P} \left\{ \max_{m \in \mathbb{Z}_M \setminus \{0\}} \left| \sum_{b \in B} e^{2\pi ibm/M} \right| < C \log M \right\} \geq 1 - \frac{1}{M \sqrt{8\tau}}.
\]

Proof. Let us apply Lemma 2.2.9 with \( A = \mathbb{Z}_M \). Then, since \( ||\mathbb{Z}_M||_u = 0 \) and \( \sigma^2 = \frac{|A|}{M^2 \tau} (1 - \tau) = \frac{\tau(1-\tau)}{M^2} \), for any \( \lambda > 0 \) we obtain

\[
\mathbb{P} \left\{ ||B||_u \geq \lambda \sqrt{\frac{\tau(1-\tau)}{M}} \right\} \leq 4M \max \left\{ e^{-\frac{\lambda^2}{8}}, e^{-\frac{\lambda\sqrt{\tau(1-\tau)}}{2\sqrt{2}}} \right\}.
\]
Then, by choosing \( \lambda = \frac{C}{\sqrt{\pi(1-\tau)}} \sqrt{M} \log M \) with a constant \( C > 4\sqrt{2} \), we ensure that
\[
4M \max \left\{ e^{-\frac{\lambda^2}{4}}, e^{-\frac{\lambda \sqrt{1-\tau}}{2\sqrt{2}M}} \right\} = \max \left\{ e^{-\frac{C^2 M \log^2 M}{2\pi(1-\tau)} + \log(4M)}, e^{-\frac{C \log M}{2\sqrt{2}} + \log(4M)} \right\} = \frac{1}{M^{2\sqrt{2} - 2}}.
\]
Thus, we obtain that
\[
P \{ \| B \|_u \geq C \log M \} \leq \frac{1}{M^{2\sqrt{2} - 2}},
\]
and \( \| B \|_u = \max_{m \in \mathbb{Z}_M \setminus \{0\}} |(J_1 B)(m)| = \max_{m \in \mathbb{Z}_M \setminus \{0\}} \left| \sum_{h \in B} e^{2\pi i bm/M} \right| \), which concludes the proof.

\[\square\]

### 2.3 Expander graphs

Let us consider an undirected \( d \)-regular graph \( G = (V, E) \), that is, a graph where each vertex is incident to exactly \( d \) edges. The number of vertices \( |V| \) is denoted by \( n \). For any \( S, T \subset V \), we denote the set of edges between \( S \) and \( T \) by \( E(S, T) = \{(u, v), \text{s.t. } u \in S, v \in T, (u, v) \in E\} \).

**Definition 2.3.1.** Let \( G = (V, E) \) be an undirected graph.

1. The **edge boundary** of a set \( S \subset V \) is given by \( \partial S = E(S, V \setminus S) \).
2. The **(edge) expansion ratio** of \( G \) is defined as
\[
h(G) = \min_{S \subset V, |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}.
\]
3. A sequence of \( d \)-regular graphs \( \{G_i\}_{i \in \mathbb{N}} \) of strictly increasing size \( |V(G_i)| \) is called a family of **expander graphs** if there exists \( \varepsilon > 0 \) such that \( h(G_i) \geq \varepsilon \), for all \( i \in \mathbb{N} \).

Some explicit constructions of families of expander graphs can be found, for example, in [40, 53, 81].

The **adjacency matrix** \( A = A(G) \) of \( G \) with \( |V(G)| = n \) is an \( n \times n \) matrix whose \((u, v)\) entry is equal to the number of edges in \( G \) connecting vertices \( u \) and \( v \). Being real and symmetric, \( A(G) \) has \( n \) real eigenvalues \( d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). We refer to the set \( \{\lambda_i\}_{i=1}^n \) of eigenvalues of \( A(G) \) as the **spectrum** of the graph \( G \). The spectrum encodes information about the connectivity of the graph. For example, \( G \) is connected if and only if \( \lambda_1 > \lambda_2 \). The following theorem shows how the second eigenvalue is related to the expansion ratio of the graph [2, 40].

**Theorem 2.3.2.** **(Cheeger inequality)** Let \( G \) be a \( d \)-regular graph with the spectrum \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and set \( \lambda = \max\{|\lambda_2|, |\lambda_n|\} \). Then
\[
d - \lambda \leq h(G) \leq \sqrt{2d(d - \lambda)}.
\]

The value \( \text{spg}(G) = \frac{d - \lambda}{d} \) is also known as the **spectral gap** of \( G \). Theorem 2.3.2 shows that a \( d \)-regular graph has an expansion ratio bounded away from zero if and only if its spectral gap is bounded away from zero. So, in order to construct graphs with big expansion ratios, one needs to construct graphs with big spectral gaps. The following theorem shows how big the spectral gap of a graph can be [40].
Theorem 2.3.3. (Allon-Boppana). For every $d$-regular graph with $|V(G)| = n$, we have
\[
\lambda = \max\{|\lambda_2|, |\lambda_n|\} \geq 2\sqrt{d-1} - o(n). 
\]

Graphs $G$ for which this bound is tight, that is, $\text{spg}(G) \leq \frac{1}{4}(d - 2\sqrt{d - 1})$, are called Ramanujan graphs. Their existence for any degree $d \in \mathbb{N}$, $d \geq 4$, has been proved in a 2013 paper by Marcus, Spielman, and Srivastava [56].

To use graphs for phase retrieval with polarization, the following version of [38, Lemma 5.2.] is useful, see also [1].

Lemma 2.3.4. Let $G$ be a $d$-regular graph. For all $\varepsilon \leq \frac{\text{spg}(G)}{6}$, the graph obtained by removing any $\varepsilon n$ vertices from $G$ has a connected component of size at least
\[
\left(1 - \frac{2\varepsilon}{\text{spg}(G)}\right) n.
\]
Investigation of geometric properties of frames that reflect their “quality” plays a crucial role in different signal processing problems, such as compressive sensing [33], quantization [54, 20], and phase retrieval problem [17, 1, 65], which is discussed in details in Chapter 4. Such geometric properties are sufficiently well-studied for randomly generated Gaussian frames with independent frame vectors. Moreover, it is often the case that Gaussian frames have properties optimal for applications with high probability. At the same time, the concrete application for which a signal processing problem is studied usually dictates the structure of the frame used to represent a signal. This motivates the study of structured, application relevant, frames.

In this chapter, we discuss geometric properties of Gabor frames, which arise naturally in many signal processing applications. We start with a short overview of known results on spark, coherence, and restricted isometry property of Gabor frames, and then investigate frame order statistics and optimal frame bounds for Gabor frames with a random window. We also compare the results obtained for Gabor frames with the respective results for random frames with independent entries, such as Gaussian frames. This comparison suggests that Gabor frames with a random window are nearly as “good” as Gaussian frames.

3.1 Overview of previously obtained results

In this section we review some previously obtained results on properties of Gabor frames with a random window and compare them to the analogous results of random frames with independent Gaussian frame vectors.

3.1.1 Spark

While the maximal number of linearly independent vectors in a frame is always equal to the dimension of the ambient space, the minimal number of linearly dependent vectors is an important property in frame theory.

Definition 3.1.1. The spark of a frame $\Phi \subset \mathbb{C}^M$ is the cardinality of its smallest linearly dependent subset. The frame is called full spark if its spark is equal to $M + 1$, that is, the frame vectors are in general linear position.

Chapter 3

Geometric properties of Gabor frames
In communication systems, when a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is used for encoding of a signal $x \in \mathbb{C}^M$, the frame spark is of major importance. If data is transmitted over an erasure channel, some of the transmitted coefficients may be lost. If only a subset $\{\langle x, \varphi_j \rangle\}_{j \in J}, J \subset \{1, \ldots, N\}$, of coefficients is received, then the original signal $x$ can still be recovered if and only if $\{\varphi_j\}_{j \in J}$ is also a frame for $\mathbb{C}^M$. If a frame $\Phi$ is full spark, removal of any $L \leq |\Phi| - M$ vectors from $\Phi$ leaves a frame, which enables reconstruction even for very unstable channels.

The following result was shown for $M$ prime in [51] and for $M$ composite in [55].

**Theorem 3.1.2.** Consider $\Lambda \subset \mathbb{Z}^M \times \mathbb{Z}^M$ with $|\Lambda| \geq M$. Then, for almost all $g \in S^{M-1}$, $(g, \Lambda)$ is a full spark frame.

### 3.1.2 Coherence

One way of measuring how uniformly frame vectors are distributed in the ambient space is to focus on the angles between pairs of vectors. More precisely, one should expect that if frame vectors “cover all directions equally”, there would be no two vectors with too small an angle between them. This idea can be formalized as follows.

**Definition 3.1.3.** Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a unit norm frame. The coherence of $\Phi$ is defined as

$$
\mu(\Phi) = \max_{j \neq k} |\langle \varphi_j, \varphi_k \rangle|.
$$

In frame theory, it is a well-known fact that for any unit norm frame $\Phi \subset \mathbb{C}^M$ with $|\Phi| = N$, we have

$$
\mu(\Phi) \geq \sqrt{\frac{N - M}{M(N - 1)}},
$$

see, for example, [75] and references therein.

The following result on the coherence of full Gabor frames is given in [67].

**Theorem 3.1.4.** Let $(g, \mathbb{Z}_M \times \mathbb{Z}_M)$ be a full Gabor frame with a window $g \in \mathbb{C}^M$, such that $g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}$, where $y_m, m \in \mathbb{Z}_M$, are independent and uniformly distributed on $[0, 1)$. Then for $\alpha = O(\log M)$

$$
\mu(g, \mathbb{Z}_M \times \mathbb{Z}_M) < \frac{\alpha}{\sqrt{M}},
$$

with high probability.

As Theorem 3.1.4 shows, the coherence of a full Gabor frame is close (up to the factor $\alpha$) to the lower coherence bound $\sqrt{\frac{M^2 - M}{M(M^2 - 1)}} = \frac{1}{\sqrt{M+1}}$ with high probability.

### 3.1.3 Restricted isometry property

While the set of frame vectors is usually redundant and thus cannot be an orthonormal system, one may ask how close it is to one. The following notion characterizes matrices that are nearly orthogonal, at least when operating on sparse vectors.
Definition 3.1.5. Let $\Phi \in \mathbb{C}^{M \times N}$ be a matrix and fix $s < N$. The restricted isometry constant $\delta_s$ of $\Phi$ is the smallest number for which the double inequality

$$(1 - \delta_s) ||x||_2^2 \leq ||\Phi x||_2^2 \leq (1 + \delta_s) ||x||_2^2$$

is satisfied for all $s$-sparse vectors $x \in \mathbb{C}^N$, that is, such that $|\text{supp}(x)| \leq s$.

It has been shown that, under the condition that the restricted isometry constant of a matrix $\Phi$ is sufficiently close to zero, a variety of recovery algorithms reconstruct every $s$-sparse vector $x$ from its “compressed” measurements $y = \Phi x$ [32, 33]. Among these algorithms are $\ell_1$ norm minimization [16], orthogonal matching pursuit [84], and hard thresholding pursuit [31].

While in practice the structure of the measurement matrix $\Phi$ is often prescribed, all matrices with optimal restricted isometry constant known so far are random matrices with independent entries [33]. For the time-frequency structured matrices, whose columns are vectors of a full Gabor frame $(g, \mathbb{Z}_M \times \mathbb{Z}_M)$ with some $g \in \mathbb{C}^M$, the following result is shown in [48].

Theorem 3.1.6. Let $g \in \mathbb{C}^M$ be a random vector with independent, mean-zero, variance one, Gaussian entries. Consider the $M \times M^2$ synthesis matrix $\Phi$ of the full Gabor frame $(g, \mathbb{Z}_M \times \mathbb{Z}_M)$. If, for $s \in \mathbb{N}$ and $\delta, \eta \in (0, 1)$,

$$M \geq c\delta^{-2}s \max \{ (\log^2 s)(\log^2 M), \log(\eta^{-1}) \}$$

then with probability at least $1 - \eta$ the restricted isometry constant $\delta_s$ of $\Phi$ satisfies $\delta_s \leq \delta$. Here $c > 0$ is a numerical constant.

Note that the bound on the number $M$ of rows obtained in Theorem 3.1.6 is linear in the sparsity level $s$ up to the logarithmic factor. In other words, this shows that random time-frequency structured matrices have close to optimal restricted isometry constant.

### 3.2 Frame order statistics (FOS)

Another way to measure how well frame vectors are distributed in space is to investigate the distribution of frame coefficients. If frame vectors are spaced sufficiently uniformly in $\mathbb{C}^M$, one should expect that for a signal $x \in \mathbb{S}^{M-1}$, only few frame coefficients can be very large or very small in absolute value. Roughly speaking, this means that for each one-dimensional subspace of $\mathbb{C}^M$, there are not too many frame vectors that are almost colinear or almost orthogonal to it. To formalize this idea we introduce the following definition.

**Definition 3.2.1.** Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{C}^M$.

(i) For $\alpha \leq N$, the $\alpha$-smallest frame order statistics of $\Phi$ is given by

$$S_{FOS}(\Phi, \alpha, x) = \max_{J \subseteq \{1, \ldots, N\}, |J| \geq \alpha} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$
(ii) For $\beta \leq N$, the $\beta$-largest frame order statistics of $\Phi$ is given by
\[
L_{\text{FOS}}(\Phi, \beta, x) = \min_{J \subseteq \{1, \ldots, N\}, \ |J| \geq \beta} \max_{j \in J} |\langle x, \varphi_j \rangle|.
\]
As follows from the definition, if we delete $\lfloor N - \alpha \rfloor$ smallest and $\lfloor N - \beta \rfloor$ largest
in modulus frame coefficients, then the remaining ones satisfy
\[
S_{\text{FOS}}(\Phi, \alpha, x) \leq |\langle x, \varphi_j \rangle| \leq L_{\text{FOS}}(\Phi, \beta, x).
\]
If $\Phi$ is a random frame, then $S_{\text{FOS}}(\Phi, \alpha)$ and $L_{\text{FOS}}(\Phi, \beta)$ provide bounds on the
$\lfloor N - \alpha \rfloor$-th and $\lfloor \beta \rfloor$-th order statistics of the vector of frame coefficients, respectively.
In signal processing, one is often looking for a sparse representation of a signal, where only few frame coefficients are nonzero. At the same time, frames for which
the vector of frame coefficients is essentially “flat”, that is, ones with sufficiently
large $S_{\text{FOS}}$ and small $L_{\text{FOS}}$, are also of interest in many applications. For instance,
such frames are used in quantization and allow to reduce quantization errors [54, 20].
Frame order statistics also play an important role in stability and robustness analysis
of the phase retrieval problem [1, 65]. We discuss this in more details in Chapter 4.

For Gabor frames with random windows, we show the following bounds on their
frame order statistics. This result can be also found in [65].

**Theorem 3.2.2.** Fix $x \in \mathbb{S}^{M-1}$ and consider a Gabor frame $\Phi_\Lambda = (g, \Lambda)$ with
$\Lambda \subseteq \mathbb{Z}_M \times \mathbb{Z}_M$ and a random window $g$ uniformly distributed on the unit sphere
$\mathbb{S}^{M-1}$. Then the following holds.

(a) For any $c > 0$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have
\[
S_{\text{FOS}}(\Phi_\Lambda, |\Lambda|(1 - c^2 + kc), x) \geq \frac{c}{\sqrt{M}}.
\]

(b) For any $C > 1$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have
\[
L_{\text{FOS}}(\Phi_\Lambda, |\Lambda| \left(1 - \frac{8}{\pi} e^{-\frac{C^2}{\pi}} + \frac{k^2}{\sqrt{\pi}} e^{-\frac{C^2}{\pi}}\right), x) \leq \frac{C}{\sqrt{M}}.
\]

**Proof.** (a) For $x \in \mathbb{S}^{M-1}$ fixed, we set
\[
G_\delta(x) = \{\varphi \in \mathbb{S}^{M-1} \subset \mathbb{C}^M, \text{ s.t. } |\langle x, \varphi \rangle| < \delta\}.
\]
We are interested in the distribution of the random variable
\[
Z_x = |G_\delta(x) \cap (g, \Lambda)| = \sum_{\lambda \in \Lambda} 1_{\{\pi(\lambda)g \in G_\delta(x)\}},
\]
where $1_{\{\varphi \in G_\delta(x)\}}$ is the characteristic function of the event $\{\varphi \in G_\delta(x)\}$. In words,
$Z_x$ is the number of small measurements of the fixed signal $x$ with respect to the
Gabor frame $(g, \Lambda)$ with a random window $g$.

First, note that for each $\lambda \in \Lambda$, $\pi(\lambda)g$ is also uniformly distributed on $\mathbb{S}^{M-1}$, that
is, the random vectors $\pi(\lambda)g$, $\lambda \in \Lambda$, have identical distribution. Indeed, consider
a random vector $h$, such that $h(m) \sim i.i.d. \mathcal{CN}(0, \frac{1}{M})$. Then, since random vector
that is, words, we can write

\[ g = h/\|h\|_2 \]  

Using equations (3.1) and (3.3), we obtain

\[ \text{small, we obtain} \]

\[ \{ |\langle M \rangle| \} = \text{Var}(e^{2\pi i m t/M} h(m)) = Var(h(m)) = \frac{1}{M}. \]

Thus the distribution of \( h \) is preserved by both modulation and time shift, and \( \pi(\lambda) g = \pi(\lambda) h/\|\pi(\lambda) h\|_2 \) is also uniformly distributed on the unit sphere \( S^{M-1} \).

Since \( \pi(\lambda) g \) has the same distribution as \( g \), for all \( \lambda \in \Lambda \) we have

\[ \mathbb{P}\{ |\langle x, \pi(\lambda) g \rangle| < \delta \} = \mathbb{P}\{ |\langle x, g \rangle| < \delta \}. \]

Let \( R \) be a unitary matrix, such that \( x = Re_1 \), where \( e_1 = (1,0,0,\ldots,0)^T \) is the first vector of the standard basis of \( \mathbb{C}^M \). Then \( |\langle x, g \rangle| = |\langle Re_1, g \rangle| = |\langle e_1, R^* g \rangle| \). By the rotational symmetry of the distribution of \( g \), we obtain

\[ \mathbb{P}\{ |\langle x, g \rangle| < \delta \} = \mathbb{P}\{ |\langle e_1, g \rangle| < \delta \} = \mathbb{P}\{ |g(0)| < \delta \}. \]

Let us identify the complex unit sphere \( S^{M-1} \subset \mathbb{C}^M \) with the real unit sphere \( S^{2M-1}_F \subset \mathbb{R}^M \), using the map \( \mathcal{I} : S^{M-1} \to S^{2M-1}_F \) given by

\[ \mathcal{I}(z_0, \ldots, z_{M-1}) = (\Re(z_0), \Im(z_0), \ldots, \Re(z_{M-1}), \Im(z_{M-1})). \]  

Since \( g \) is uniformly distributed on \( S^{M-1} \), \( \tilde{g} = \mathcal{I}(g) \) is uniformly distributed on \( S^{2M-1}_F \). Thus

\[ \mathbb{P}\{ |g(0)| < \delta \} = \mathbb{P}\{ |g(1)^2 + \tilde{g}(1)^2 < \delta^2 \} = \frac{S_{<\delta}}{S_1}, \]

where \( S_1 = \frac{2^M}{(M-1)!} \) is the surface area of \( S^{2M-1}_F \), and \( S_{<\delta} \) is the surface area of the set \( \{ z \in S^{2M-1}_F \text{ s.t. } z_0^2 + z_1^2 < \delta^2 \} \).

\[ S_{<\delta} = \int_{-\delta}^\delta \int_{-\delta}^{\sqrt{\delta^2-z_0^2}} \frac{2\pi^{M-1}}{(M-2)!} \left(1 - \frac{z_0^2 + z_1^2}{\delta^2}ight)^{M-3/2} dz_1 dz_0 \leq \frac{2\pi^M \delta^2}{(M-2)!}, \]

that is, \( \mathbb{P}\{ |g(0)| < \delta \} \leq \delta^2(M - 1) \). Now, setting \( \delta = \frac{c}{\sqrt{M}} \) for \( c > 0 \) sufficiently small, we obtain

\[ \mathbb{P}\left\{ |\langle x, g \rangle| < \frac{c}{\sqrt{M}} \right\} \leq c^2. \]  

Using equations (3.1) and (3.3), we obtain

\[ \mu = \mathbb{E}(Z_x) = |\Lambda| \mathbb{P}\left\{ |\langle x, g \rangle| < \frac{c}{\sqrt{M}} \right\} \leq |\Lambda| c^2. \]
Similarly, using (3.3), for the variance of $Z_x$ we obtain

$$
\sigma^2 = \text{Var}(Z_x) = \mathbb{E}(Z_x^2) - (\mathbb{E}(Z_x))^2 \leq \mathbb{E}(Z_x^2) = \mathbb{E} \left( \left( \sum_{\lambda \in \Lambda} 1_{\{\pi(\lambda)g \in G_{\sqrt{\pi}}(x)\}} \right)^2 \right)
$$

$$
= \mathbb{E} \left( \sum_{\lambda \in \Lambda} 1_{\{\pi(\lambda)g \in G_{\sqrt{\pi}}(x)\}}^2 + \sum_{(\lambda_1, \lambda_2) \in \Lambda^2, \lambda_1 \neq \lambda_2} 1_{\{\pi(\lambda_1)g \in G_{\sqrt{\pi}}(x)\}} \cdot 1_{\{\pi(\lambda_2)g \in G_{\sqrt{\pi}}(x)\}} \right)
$$

$$
= \sum_{\lambda \in \Lambda} \mathbb{P} \left\{ \pi(\lambda)g \in G_{\sqrt{\pi}}(x) \right\} + \sum_{(\lambda_1, \lambda_2) \in \Lambda^2, \lambda_1 \neq \lambda_2} \mathbb{P} \left\{ \pi(\lambda_1)g, \pi(\lambda_2)g \in G_{\sqrt{\pi}}(x) \right\}
$$

$$
\leq (|\Lambda| + (|\Lambda|^2 - |\Lambda|)) \mathbb{P} \left\{ |\langle x, g \rangle| < \frac{c}{\sqrt{M}} \right\} \leq c^2 |\Lambda|^2. \quad (3.5)
$$

That is, $\sigma \leq c|\Lambda|$. Then, using Chebychev inequality and bounds (3.4) and (3.5), we have the following estimate. For any $k > 0$,

$$
\mathbb{P} \{ Z_x \geq |\Lambda|(c^2 + kc) \} \leq \mathbb{P} \{ Z_x \geq \mu + k\sigma \} \leq \mathbb{P} \{ |Z_x - \mu| \geq k\sigma \} \leq \frac{1}{k^2}.
$$

In other words, if we delete $|\Lambda|(c^2 + kc)$ smallest phaseless measurements, with probability at least $1 - \frac{1}{k^2}$, for the remaining measurements we would have $|\langle x, \pi(\lambda)g \rangle| \geq \frac{c}{\sqrt{M}}$. Thus

$$
S_{\text{FOS}} (\Phi_\Lambda, |\Lambda|(c^2 + kc), x) \geq \frac{c}{\sqrt{M}}.
$$

This concludes the proof of (a).

The proof of (b) follows essentially the same steps. Let $C > 1$ be a constant and consider the random variable

$$
U_x = \sum_{\lambda \in \Lambda} 1_{\{|\langle x, \pi(\lambda)g \rangle| > C/\sqrt{M}\}}.
$$

Since, for each $\lambda \in \Lambda$, $\pi(\lambda)g$ has the same, uniform on $S^{M-1}$, distribution, we have $\mathbb{P} \{ |\langle x, \pi(\lambda)g \rangle| > C/\sqrt{M} \} = \mathbb{P} \{ |\langle e_1, g \rangle| > C/\sqrt{M} \}$, for all $\lambda \in \Lambda$. As above, we can use the rotation symmetry of the distribution of $g$ to obtain

$$
\mathbb{P} \{ |\langle x, g \rangle| > C/\sqrt{M} \} = \mathbb{P} \{ |\langle e_1, g \rangle| > C/\sqrt{M} \} = \mathbb{P} \{ |g(0)| > C/\sqrt{M} \}.
$$

Using the map $\mathcal{I} : S^{M-1} \rightarrow S^{2M-1}_R$ defined in (3.2), for $\tilde{g} = \mathcal{I}(g)$ we obtain

$$
\mathbb{P} \left\{ |g(0)| > \frac{C}{\sqrt{M}} \right\} = \mathbb{P} \left\{ \tilde{g}(0)^2 + \tilde{g}(1)^2 > \frac{C^2}{M} \right\} = \frac{S_{\geq C/\sqrt{M}}}{S_1},
$$

where $S_1 = \frac{2e^M}{(M-1)}$ is the surface area of $S^{2M-1}_R$, and $S_{\geq C/\sqrt{M}}$ is the surface area of
the set \( \{ z \in \mathbb{S}^{2M-1}_R, \text{ s.t. } z_0^2 + z_1^2 > \frac{C^2}{M} \} \).

\[
S_{> \sqrt{M}} = \left\{ z_0 \leq 1 \int_{|z_0|<1} \left( \frac{2\pi^{M-1}}{(M-2)!} \sqrt{\frac{1-z_0^2}{z_0}} d\pi_0 \right)^2 d\pi_0 \right\}
= 8 \frac{2\pi^{M-1}}{(M-2)!} \frac{M}{2\pi^2} \int_{0}^{\sqrt{M}} \left( 1 - z_0^2 - z_1^2 \right)^{2M-3} d\pi_0 d\pi_0.
\]

\[
S_{> \sqrt{M}} \leq 16 \pi^{M-1} \sqrt{M} \int_{0}^{\sqrt{M}} \left( 1 - z_0^2 - z_1^2 \right)^{2M-3} d\pi_0 d\pi_0.
\]

Similarly, for the variance of \( S_{> \sqrt{M}} \), we obtain

\[
\sigma^2 = \text{Var}(S_{> \sqrt{M}}) \leq \mathbb{E}(S_{> \sqrt{M}}) - \mathbb{E}^2(S_{> \sqrt{M}}) = \mathbb{E}\left( \sum_{\lambda \in \Lambda} \mathbb{P}\left\{ |\langle x, \pi(\lambda)g \rangle| > \frac{C}{\sqrt{M}} \right\} \right)^2.
\]

Here, we used the symmetry of the domain of integration, the fact that \( z_0^2 + z_1^2 > \frac{C^2}{M} \)
implies \( \max\{|z_0|, |z_1|\} > \frac{C}{\sqrt{2M}} \), and the inequality \( 1 - x \leq e^{-x} \).

Using the computed bound for \( S_{> \sqrt{M}} \), we obtain

\[
\mathbb{P}\left\{ |g(0)| > \frac{C}{\sqrt{M}} \right\} \leq \frac{8}{\pi} e^{-C^2/2}.
\]

Then

\[
\mu = \mathbb{E}(U_x) = \sum_{\lambda \in \Lambda} \mathbb{P}\left\{ |\langle x, \pi(\lambda)g \rangle| > \frac{C}{\sqrt{M}} \right\} \leq \frac{8}{\pi} e^{-C^2/2} |\Lambda|.
\]

Similarly, for the variance of \( U_x \) we obtain

\[
\sigma^2 = \text{Var}(U_x) \leq \mathbb{E}(U_x^2) - \mathbb{E}^2(U_x) = \mathbb{E}\left( \sum_{\lambda \in \Lambda} \mathbb{P}\left\{ |\langle x, \pi(\lambda)g \rangle| > \frac{C}{\sqrt{M}} \right\} \right)^2.
\]

That is, \( \sigma^2 \leq \frac{2\pi}{\sqrt{\pi}} e^{-\frac{C^2}{2}} |\Lambda| \). Then, using Chebychev inequality and bounds (3.6) and

\[
\mathbb{P}\left\{ U_x > |\Lambda| \left( \frac{8}{\pi} e^{-\frac{C^2}{2}} + k \frac{2\sqrt{2}}{\sqrt{\pi}} e^{-\frac{C^2}{4}} \right) \right\} \leq \mathbb{P}\{|U_x - \mu| \geq k\sigma\} \leq \frac{1}{k^2}.
\]

In other words, if we delete \( |\Lambda| \left( \frac{8}{\pi} e^{-\frac{C^2}{2}} + k \frac{2\sqrt{2}}{\sqrt{\pi}} e^{-\frac{C^2}{4}} \right) \) largest phaseless
measurements, with probability at least \( 1 - \frac{1}{k^2} \) for the remaining measurements we would
have $|\langle x, \pi(\lambda) g \rangle| \leq \frac{C}{\sqrt{M}}$. Thus

$$\mathcal{L}_{FOS}(\Phi, |\Lambda| \left(\frac{8}{\pi} e^{-\frac{C^2}{4}} + k \frac{2}{\sqrt{\pi}} e^{-\frac{C^2}{4}}\right), x) \leq \frac{C}{\sqrt{M}}.$$ 

\[\Box\]

**Remark 3.2.3.** A similar result can be shown for a Gabor frame with a window whose entries are independent Gaussian random variables. The proof in this case involves the same steps as the proof of Theorem 3.2.2.

Moreover, the only property of Gabor frames that the proof of Theorem 3.2.2 uses is that all the frame vectors have the same uniform distribution on $\mathbb{S}^{M-1}$. Thus, the same bounds on the frame order statistics hold for any random frame with not necessarily independent frame vectors uniformly distributed on $\mathbb{S}^{M-1}$.

We also note that in the real case, when we assume that window $g$ is uniformly distributed on the real unit sphere $\mathbb{S}^{M-1}_\mathbb{R} \subset \mathbb{R}^M$, and $x \in \mathbb{S}^{M-1}_\mathbb{R}$, the proof of Theorem 3.2.2 is still valid. Thus the real case version of Theorem 3.2.2 is also true.

Theorem 3.2.2 is a non-uniform result in the sense that the proven bounds hold with high probability for each individual signal $x$. At the same time, we are interested in obtaining uniform bounds, which hold with high probability for all $x \in \mathbb{S}^{M-1}$ simultaneously. We introduce the following notion of uniform frame order statistics.

**Definition 3.2.4.** Let $\Phi \subset \mathbb{S}^{M-1}$ be a unit norm frame.

(i) For $\alpha \leq N$, define the $\alpha$-smallest uniform frame order statistics\(^1\) of $\Phi$ as

$$\mathcal{S}_{uFOS}(\Phi, \alpha) = \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha, x).$$

(ii) For $\beta \leq N$, define the $\beta$-largest uniform frame order statistics of $\Phi$ as

$$\mathcal{L}_{uFOS}(\Phi, \beta) = \max_{x \in \mathbb{S}^{M-1}} \mathcal{L}_{FOS}(\Phi, \beta, x).$$

We give a uniform bound on the number of large frame coefficients with respect to a Gabor frame with a Gaussian random window in the following result, see also [65].

**Theorem 3.2.5.** Consider a Gabor frame $\Phi_\Lambda = (g, \Lambda)$ with a random Gaussian window $g$, such that $g(m) \sim i.i.d. \mathcal{CN}\left(0, \frac{1}{\sqrt{M}}\right)$. Then, for some suitably chosen numerical constants $c, c_1 > 0$,

$$\mathcal{L}_{uFOS}(\Phi_\Lambda, |\Lambda| \leq \frac{cM}{\log^4 M}) \leq \sqrt{\frac{3 \log^2 M}{2c \cdot \sqrt{M}}},$$

with probability at least $1 - e^{-c_1 \log^3 M}$.

\(^1\)The notion of the $\alpha$-smallest uniform frame order statistics has been also considered for Gaussian frames in [1]. There authors use the term *projective uniformity (PU)* instead.
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Proof. We first consider the case of a full Gabor frame. Let $\Phi = \Phi_{\mathbb{Z}_M^2 \times \mathbb{Z}_M^2}$ be an $M \times M^2$ matrix whose columns are $\pi(\lambda)g$, $\lambda \in \mathbb{Z}_M^2 \times \mathbb{Z}_M^2$. Fix $s \in \{1, \ldots, M^2\}$, and for any $x \in \mathbb{S}^{M^2}$ denote by $S_x$ the set of $\lambda \in \mathbb{Z}_M^2 \times \mathbb{Z}_M^2$ corresponding to the $s$ biggest in modulus frame coefficients of $x$ with respect to the Gabor frame $(g, \mathbb{Z}_M^2 \times \mathbb{Z}_M^2)$. Then, for the phase vector $v_x \in \mathbb{C}^{M^2}$ defined by

$$v_x(\lambda) = \begin{cases} \frac{\langle x, \pi(\lambda)g \rangle}{||x, \pi(\lambda)g||}, & \lambda \in S_x, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$x^* \Phi v_x = \sum_{\lambda \in S_x} ||x, \pi(\lambda)g||.$$

After applying the Cauchy–Schwarz inequality to $x^* \Phi v_x = \langle \Phi v_x, x \rangle$, we obtain

$$\sum_{\lambda \in S_x} |\langle x, \pi(\lambda)g \rangle| \leq ||x||_2 ||\Phi v_x||_2 = ||\Phi v_x||_2.$$

Note that $v_x$ is an $s$-sparse vector with $||v_x||_2 = \sqrt{s}$. Then, if $s = \frac{cM}{\log^4 M}$ for a suitably chosen numerical constant $c > 0$, Theorem 3.1.6 implies that

$$\frac{1}{2} ||v_x||_2^2 \leq ||\Phi v_x||_2 \leq \frac{3}{2} ||v_x||_2^2,$$

for any $s$-sparse vector $v \in \mathbb{C}^{M^2}$, with probability at least $1 - e^{-c_1 \log^3 M}$, where $c_1 > 0$ depends only on $c$. Thus

$$\sum_{\lambda \in S_x} |\langle x, \pi(\lambda)g \rangle| \leq ||\Phi v_x||_2 \leq \sqrt{\frac{3s}{2}},$$

with probability at least $1 - e^{-c_1 \log^3 M}$. It follows that with the same probability,

$$\min_{\lambda \in S_x} |\langle x, \pi(\lambda)g \rangle| \leq \sqrt{\frac{3}{2s}} = \sqrt{\frac{3 \log^2 M}{2c \sqrt{M}}}.$$

In other words, with probability at least $1 - e^{-c_1 \log^3 M}$, for any $x \in \mathbb{S}^{M^2}$ all except at most $\frac{cM}{\log^4 M} - 1$ frame coefficients are in modulus smaller than $\sqrt{\frac{3 \log^2 M}{2c \sqrt{M}}}$. Since this is true for the full Gabor frame $(g, \mathbb{Z}_M^2 \times \mathbb{Z}_M^2)$, it also holds for any its subframe $(g, \Lambda)$ with $\Lambda \subseteq \mathbb{Z}_M^2$, and the claim of the theorem follows.

$$\square$$

We note that, while Theorem 3.2.5 gives a better bound on the number of large frame coefficients than Theorem 3.2.2 (b), that is, $\frac{cM}{\log^4 M}$ instead of $c|\Lambda| \geq CM$, it gives a slightly weaker bound on the modulus of the remaining coefficients, namely, $\frac{C \log^2 M}{\sqrt{M}}$ instead of $\frac{C}{\sqrt{M}}$.

Remark 3.2.6. Note that Theorem 3.1.6 also holds whenever the window $g$ has independent mean-zero, variance one, $L$-subgaussian entries [48]. Thus, Theorem 3.2.5 can also be formulated for this larger class of Gabor windows.
We have also studied the uniform frame order statistics for random frames with independent frame vectors satisfying certain moment assumptions. To compare the obtained bounds on the frame order statistics of Gabor frames (Theorems 3.2.2 and 3.2.5) with the respective results for random frames with independent frame vectors, we state the following result here. The proof of this result is presented in Appendix A, Theorems A.2.1 and A.3.1. Theorem A.2.1 can be also found in the Master’s thesis of the author of this manuscript [72].

**Theorem 3.2.7.** Let the frame \( \Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M \) with \( M \) large enough be such that \( \varphi_j(m), j \in \{1,\ldots,N\}, m \in \mathbb{Z}_M \), are independent identically distributed centered random variables, normalized so that \( \text{Var}(\varphi_j(m)) = \frac{1}{M} \). Assume further that \( \mathbb{E}(|\varphi_i(m)|^4) \leq \frac{B}{M^2} \), for some constant \( B \geq 1 \), and \( N \geq C_0 M \log M \), for some constant \( C_0 \). Then the following holds.

(a) For each fixed \( \alpha < 1 - \frac{1}{2C_0} \),

\[
\mathcal{S}_{\text{uFOS}}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}}
\]

with probability at least \( 1 - e^{-c_1 M \log M} \), where constants \( c, c_1 > 0 \) depend only on \( B, \alpha, \) and \( C_0 \).

(b) For each fixed \( \beta < 1 - \frac{1}{2C_0} \),

\[
\mathcal{L}_{\text{uFOS}}(\Phi, \beta N) \leq \frac{K}{\sqrt{M}}
\]

with probability at least \( 1 - e^{-c_1 M \log M} \), where constants \( K, c_1 > 0 \) depend only on \( B, \beta, \) and \( C_0 \).

We note, that the class of random frames considered here is quite large, and, in particular, it includes Gaussian random frames with independent frame vectors, as well as L-subgaussian random frames (see Section 2.2 for the definition).

**Remark 3.2.8.** In the real case, when we assume that \( \Phi \subset \mathbb{R}^M \) satisfies the same moment assumptions, and \( x \in \mathbb{S}^{M-1}_\mathbb{R} \), the proof of Theorem 3.2.7 remains valid. Thus the real case version of Theorem 3.2.7 holds as well.

### 3.2.1 Numerical results

Theorem 3.2.2 shows that, for a Gabor frame \( \Phi_\Lambda \) with a random window and a fixed signal \( x \in \mathbb{S}^{M-1}_\mathbb{R} \), with high probability, the frame order statistics of \( \Phi_\Lambda \) satisfy

\[
\frac{c}{\sqrt{M}} \leq \mathcal{S}_{\text{FOS}}(\Phi_\Lambda, \alpha, x) \leq \mathcal{L}_{\text{FOS}}(\Phi_\Lambda, \beta, x) \leq \frac{C}{\sqrt{M}},
\]

for \( \alpha, \beta \geq \frac{1}{2}|\Lambda| \) and suitably chosen constants \( 0 < c < C \) depending on \( \alpha \) and \( \beta \), respectively. In other words, for a fixed signal \( x \in \mathbb{S}^{M-1}_\mathbb{R} \), absolute values of most of the frame coefficients are tightly concentrated around their expected value \( \mathbb{E}(|\langle x, \pi(\lambda)g \rangle|) = \frac{1}{\sqrt{M}} \).

This result is significant to better understand of geometric properties of Gabor frames and plays an essential role in establishing robustness guarantees for phase
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Figure 3.1: Dependence of the numerically estimated $S_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|)$ (left) and its scaled version $\sqrt{M}S_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|)$ (right) on the ambient dimension $M$, where $\Phi_\Lambda = (g, \Lambda)$ is a Gabor frame with a random uniformly distributed on $S^{M-1}$ window $g$. Numerical results on these plots suggest that $\alpha$-smallest uniform frame order statistics of $\Phi_\Lambda$ decays with $M$ as $O(\sqrt{M})$ and, furthermore, $S_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|) \geq 0.3\sqrt{M}$.

retrieval, as we discuss in Section 4.5.2. At the same time, the study of the uniform frame order statistics for Gabor frames, as well as other structured frames, remains an important task. A uniform result, similar to Theorem 3.2.7, would lead to a significant step forward in phase retrieval with Gabor frames, see, for example, discussion in Section 4.3.2 and Conjecture 4.5.4. Moreover, such a result would be important for noise-shaping quantization based on Gabor frame measurements [20].

In this section we use numerical simulations to investigate the behavior of uniform frame order statistics of Gabor frames. More precisely, we numerically test various Gabor frames $\Phi_\Lambda = (g, \Lambda)$ with a fixed $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ and a random window $g$ uniformly distributed on the unit sphere $S^{M-1}$ to obtain estimates of $S_{uFOS}(\Phi_\Lambda, \alpha|\Lambda|)$ and $L_{uFOS}(\Phi_\Lambda, \beta|\Lambda|)$.

The obtained numerical results are presented on Figures 3.1 and 3.2. They suggest that both $\alpha$-smallest uniform frame order statistics and $\beta$-largest uniform frame order statistics of $\Phi_\Lambda$ decay with $M$ as $O(\sqrt{M})$, but with different constants. For example, in the case when $\alpha = \beta = 0.9|\Lambda|$, numerical results seem to indicate that $0.3\sqrt{M} \leq S_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|) \leq L_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|) \leq 1.52\sqrt{M}$.

Based on the numerical findings, we formulate the following conjecture.

**Conjecture 3.2.9.** Consider a Gabor frame $\Phi_\Lambda = (g, \Lambda)$ with $|\Lambda| \geq C_0M(\log M)^\gamma$ (parameter $\gamma \geq 0$ to be specified), $C_0 > 0$, and a random window $g$ uniformly distributed on the unit sphere $S^{M-1}$. Then the following holds.
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Figure 3.2: Dependence of the numerically estimated $\mathcal{L}_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|)$ (left) and its scaled version $\sqrt{M}\mathcal{L}_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|)$ (right) on the ambient dimension $M$, where $\Phi_\Lambda = (g, \Lambda)$ is a Gabor frame with a random uniformly distributed on $\mathbb{S}^{M-1}$ window $g$. Here, $|\mathcal{T}| = O(M)$. Numerical results suggest that $\beta$-largest uniform frame order statistics of $\Phi_\Lambda$ decays with $M$ as $\frac{K}{\sqrt{M}}$, and, further, $\mathcal{L}_{uFOS}(\Phi_\Lambda, 0.9|\Lambda|) \leq \frac{1.52}{\sqrt{M}}$.

(a) For any $c \in (0, 1)$, there exists $\epsilon(c) \in (0, 1)$ not depending on $M$, such that with high probability

$$S_{uFOS}(\Phi_\Lambda, \epsilon(c)|\Lambda|) \geq \frac{c}{\sqrt{M}}.$$ 

(b) For any $C > 1$ there exists $\eta(C) \in (0, 1)$ not depending on $M$, such that with high probability

$$\mathcal{L}_{uFOS}(\Phi_\Lambda, \eta(C)|\Lambda|) \leq \frac{C}{\sqrt{M}}.$$ 

3.3 Singular values of Gabor analysis matrices

Frames are widely used in many signal processing problems, where a signal of interest is encoded using its frame coefficients. Among such applications are communication systems, where the frame coefficients are used to transmit a signal over the communication channel; image processing; and also tomography, speech recognition and brain imaging, where the initial signal is not available, but we have access to its measurements in the form of the frame coefficients instead. One of the key advantages of a frame compared to a basis is the redundancy of the signal representation using frame coefficients. Provided we have a control on the frame bounds, this redundancy allows, among other things, to achieve robust reconstruction of a signal from its frame coefficients that are corrupted by noise, rounding error due to quantization, or erasures.

Indeed, consider a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$. The optimal lower and upper frame bounds of $\Phi$ are given by

$$A = \min_{x \in \mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 = \min_{x \in \mathbb{S}^{M-1}} ||\Phi^* x||_2^2 = \sigma^2_{\min}(\Phi^*),$$

$$B = \max_{x \in \mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 = \max_{x \in \mathbb{S}^{M-1}} ||\Phi^* x||_2^2 = \sigma^2_{\max}(\Phi^*),$$

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where \( \Phi \) also denotes the synthesis matrix of the frame \( \Phi \), which has frame vectors \( \varphi_j, j \in \{1, \ldots, N\} \), as its columns.

Let \( c \in \mathbb{C}^N \) be a vector of noisy frame coefficients of a signal \( x \in \mathbb{C}^M \) with respect to the frame \( \Phi \). That is,

\[
c = \Phi^* x + \delta,
\]
where \( \delta \in \mathbb{C}^N \) is a noise vector. Then an estimate \( \hat{x} \) of the initial signal \( x \) can be obtained from its noisy measurements \( c \) using the standard dual frame of \( \Phi \). More precisely, we have

\[
\hat{x} = (\Phi \Phi^*)^{-1} \Phi c = x + (\Phi \Phi^*)^{-1} \Phi \delta.
\]

Thus, for the reconstruction error we have

\[
||\hat{x} - x||_2^2 \leq ||(\Phi \Phi^*)^{-1} \Phi \delta||_2^2 = \frac{||\delta||_2^2}{\sigma_{\text{min}}^2(\Phi^*)}.
\]

Moreover, if we know a bound on the signal to noise ratio \( \text{SNR} = \frac{||\Phi^* x||_2}{||\delta||_2} \) for the channel used, then the norm of the reconstruction error \( ||(\Phi \Phi^*)^{-1} \Phi \delta||_2 \) compares to the norm of the initial signal \( ||x||_2 \) as

\[
\frac{||[(\Phi \Phi^*)^{-1} \Phi \delta]||_2}{||x||_2} \leq \frac{\text{Cond}(\Phi^*)}{\text{SNR}}.
\]

Here,

\[
\text{Cond}(\Phi^*) = \sup_{x \in \mathbb{C}^M \setminus \{0\}} \sup_{\delta \in \mathbb{C}^N \setminus \{0\}} \frac{||[(\Phi \Phi^*)^{-1} \Phi \delta]||_2}{||x||_2} = \frac{\sigma_{\text{max}}(\Phi^*)}{\sigma_{\text{min}}(\Phi^*)} = \frac{\sqrt{B}}{\sqrt{A}}.
\]

That is, \( \text{Cond}(\Phi^*) \) is equal to the condition number of the analysis matrix of the frame \( \Phi \).

Thus, frame bounds indicate the "quality" of a frame in the sense of the robustness of the reconstruction of an initial signal from its noisy frame coefficients. In the case when frame bounds of \( \Phi \) are sufficiently close to each other, that is, when \( \text{Cond}(\Phi^*) \) is not too large, we call the frame \( \Phi \) well-conditioned.

Before we study the case when \( \Phi \) is a Gabor frame, we include here a short overview of the results on the singular values for random frames with independent entries. The largest singular value of the analysis matrix \( \Phi^* \) of a random frame with independent entries can be estimated using Latala’s theorem [49]. The following result implies that, with high probability, \( \sigma_{\text{max}}(\Phi^*) = O\left(\sqrt{\frac{N}{M}}\right) \).

**Theorem 3.3.1.** [49] Let \( \Phi^* \in \mathbb{C}^{N \times M}, N > M, \) be a random matrix with entries \( \varphi_j(m), j \in \{1, \ldots, N\}, m \in \mathbb{Z}_M, \) that are independent identically distributed centered random variables, normalized so that \( \text{Var}(\varphi_j(m)) = \frac{1}{M} \). Assume further that \( \mathbb{E}(||\varphi_j(m)||_4^4) \leq \frac{B}{M^2} \) for some constant \( B > 1 \). Then there exists a constant \( C > 0 \) depending only on \( B \), such that

\[
\mathbb{E}\left(\sigma_{\text{max}}(\Phi^*)\right) \leq C \sqrt{\frac{N}{M}}.
\]
The following optimal estimate of the smallest singular value of the analysis matrix for a random subgaussian frame with independent entries is due to Rudelson and Vershynin [70].

**Theorem 3.3.2.** [70] Let \( \Phi^* \in \mathbb{C}^{N \times M}, N > M, \) be a random matrix with entries \( \varphi_j(m), j \in \{1, \ldots, N\}, m \in \mathbb{Z}_M, \) that are independent identically distributed \( L \)-subgaussian random variables with zero mean, normalized so that \( \text{Var}(\varphi_j(m)) = \frac{1}{M} \). Then, for any \( \varepsilon \geq 0, \)

\[
\mathbb{P}\left\{ \sigma_{\min}(\Phi^*) > \varepsilon \left( \sqrt{\frac{N}{M}} - \sqrt{\frac{M - 1}{M}} \right) \right\} \geq 1 - (C\varepsilon)^{N-M+1} + c^N,
\]

where constants \( C > 0 \) and \( c \in (0, 1) \) depend only on \( L \).

We note that the estimate given by this result is tight also for square matrices, that is, when \( N = M \).

To the best of our knowledge, singular values of the analysis matrix \( \Phi^*_\Lambda \) of a Gabor frame with random window and general \( \Lambda, |\Lambda| > M, \) were not studied before. At the same time, the following bounds on the singular values of the synthesis matrix \( \Phi \) of a Gabor system \( (g, \Lambda) \) with a Steinhaus window \( g \) and \( |\Lambda| = O\left(\frac{M}{\log(M)}\right) \) have been established in [66].

**Theorem 3.3.3.** [66] Let \( g \) be a Steinhaus window, that is, \( g(j) = \frac{1}{\sqrt{M}} e^{2\pi i y_j}, j \in \mathbb{Z}_M, \) with \( y_j \) independent uniformly distributed on \([0, 1)\). Consider a Gabor system \((g, \Lambda)\) and let \( \varepsilon, \delta \in (0, 1) \). Suppose further that

\[
|\Lambda| \leq \frac{\delta^2 M}{4e(\log(|\Lambda|/\varepsilon) + c)},
\]

where \( c = \log(e^2/(4(e - 1))) \approx 0.0724. \) Then \( \|I_\Lambda - \Phi^*_\Lambda \Phi_\Lambda\|_2 \leq \delta \) with probability at least \( 1 - \varepsilon \).

In other words the minimal and maximal singular values of \( \Phi_\Lambda \) satisfy

\[
1 - \delta \leq \sigma_{\min}^2(\Phi_\Lambda) \leq \sigma_{\max}^2(\Phi_\Lambda) \leq 1 + \delta \quad \text{with probability at least } 1 - \varepsilon.
\]

Our aim in this section is to obtain estimates on the optimal frame bounds of Gabor frames with random windows for various choices of \( \Lambda, \) and to investigate the dependence of the optimal frame bounds on the structure and cardinality of \( \Lambda. \) We start with the following observation.

**Proposition 3.3.4.** Let \((g, \Lambda)\) be a Gabor system with \( \Lambda = F \times \mathbb{Z}_M \) for some \( F \subset \mathbb{Z}_M, F \neq \emptyset, \) and a window \( g \in \mathbb{C}^M. \) Then \((g, \Lambda)\) is a frame if and only if \( \min_{m \in \mathbb{Z}_M}\{||g_{F_m}||_2\} \neq 0, \) where \( g_{F_m} \) is the restriction of the vector \( g \) to the set of coefficients \( F_m = \{m - k\}_{k \in F} \subset \mathbb{Z}_M. \)

Moreover, in this case the optimal lower and upper frame bounds for \((g, \Lambda)\) are \( M \min_{m \in \mathbb{Z}_M}\{||g_{F_m}||_2\} \) and \( M \max_{m \in \mathbb{Z}_M}\{||g_{F_m}||_2\}, \) respectively.

**Proof.** Consider the matrix \( \Phi_\Lambda \in \mathbb{C}^{M \times |F|^M} \) corresponding to the synthesis operator of the Gabor system \((g, \Lambda), \) where \( \Lambda = F \times \mathbb{Z}_M \) with \( F \subset \mathbb{Z}_M, F \neq \emptyset, \) and \( g \in \mathbb{C}^M. \)
That is, the vectors $\pi(\lambda)g$, $\lambda \in \Lambda$, are the columns of the matrix $\Phi_\Lambda$. Then consider the matrix $\Phi_\Lambda \Phi_\Lambda^*$ corresponding to the frame operator of $(g, \Lambda)$.

$$
\Phi_\Lambda \Phi_\Lambda^*(m_1, m_2) = \sum_{\lambda \in \Lambda} (\pi(\lambda)g(m_1) \overline{(\pi(\lambda)g(m_2)})
= \sum_{k \in F} \sum_{\ell \in \mathbb{Z}_M} e^{2\pi i (m_1 - m_2)/M} g(m_1 - k) \overline{g(m_2 - k)}
= \sum_{k \in F} g(m_1 - k) \overline{g(m_2 - k)} \sum_{\ell \in \mathbb{Z}_M} e^{2\pi i (m_1 - m_2)/M},
$$

for $m_1, m_2 \in \mathbb{Z}_M$. Then, since $\sum_{\ell \in \mathbb{Z}_M} e^{2\pi i (m_1 - m_2)/M} = 0$ in the case $m_1 \neq m_2$, and $\sum_{\ell \in \mathbb{Z}_M} e^{2\pi i (m_1 - m_2)/M} = M$ for $m_1 = m_2$, we obtain

$$
\Phi_\Lambda \Phi_\Lambda^*(m_1, m_2) = \begin{cases} 0, & m_1 \neq m_2 \\ M \sum_{k \in F} |g(m_1 - k)|^2, & m_1 = m_2. \end{cases}
$$

That is, $\Phi_\Lambda \Phi_\Lambda^*$ is a diagonal matrix and, thus, the set $\{\sigma_m(\Phi_\Lambda^*)\}_{m \in \mathbb{Z}_M}$ of the singular values of the matrix $\Phi_\Lambda^*$, corresponding to the analysis operator of $(g, \Lambda)$, is equal to the set $\{\sqrt{M} ||g_{F_m}||_2 \}_{m \in \mathbb{Z}_M}$, where $F_m = \{m - k\}_{k \in F} \subset \mathbb{Z}_M$ and $g_S$ denotes the restriction of the vector $g$ to a set of coefficients $S \subset \mathbb{Z}_M$.

In particular, $(g, \Lambda)$ is a frame if and only if all the diagonal entries of $\Phi_\Lambda \Phi_\Lambda^*$ are nonzero, that is, if and only if $\min_{m \in \mathbb{Z}_M} \{|g_{F_m}||_2\} \neq 0$. Moreover, we have

$$
\sigma_{\text{min}}(\Phi_\Lambda^*) = \min_{m \in \mathbb{Z}_M} \sigma_m(\Phi_\Lambda^*) = \sqrt{M} \min_{m \in \mathbb{Z}_M} \{|g_{F_m}||_2\},
\sigma_{\text{max}}(\Phi_\Lambda^*) = \max_{m \in \mathbb{Z}_M} \sigma_m(\Phi_\Lambda^*) = \sqrt{M} \max_{m \in \mathbb{Z}_M} \{|g_{F_m}||_2\}.
$$

That is, $M \min_{m \in \mathbb{Z}_M} \{|g_{F_m}||_2\}$ and $M \max_{m \in \mathbb{Z}_M} \{|g_{F_m}||_2\}$ are the optimal lower and upper frame bounds for $(g, \Lambda)$, respectively. \hfill \Box

We note that an analogous result is true for the the case when the considered set $\Lambda$ is of the form $\Lambda = \mathbb{Z}_M \times F$, for some $F \subset \mathbb{Z}_M$. Indeed, let $W_M = \frac{1}{\sqrt{M}} \{e^{-2\pi i k t/M}\}_{k, t \in \mathbb{Z}_M}$ be the normalized Fourier matrix, and consider the Gabor frame $(g, \Lambda')$ with a window $g$ and $\Lambda' = (-F) \times \mathbb{Z}_M$. Since $W_M M_k T_k W_M^* g = e^{2\pi i k t/M} M_k T_k W_M^* g$, we have

$$
W_M \Phi_{(g, \Lambda')} \Phi_{(g, \Lambda')}^* W_M^*(m_1, m_2) = \sum_{(k, t) \in \Lambda} M_{-k} T_k W_M g(m_1) \overline{M_{-k} T_k W_M g(m_2)} = \Phi_{(W_M g, \Lambda')}^* \Phi_{(W_M g, \Lambda')}(m_1, m_2).
$$

That is, $W_M \Phi_{(g, \Lambda')} \Phi_{(g, \Lambda')}^* W_M^* \Phi_{(W_M g, \Lambda')}^* = \Phi_{(W_M g, \Lambda')}^*$. Thus,

$$
\sigma_{\text{min}}(\Phi_{(W_M g, \Lambda')}) = \sigma_{\text{min}}(\Phi_{(g, \Lambda')}),
\sigma_{\text{max}}(\Phi_{(W_M g, \Lambda')}) = \sigma_{\text{max}}(\Phi_{(g, \Lambda')}).
$$

**Example 3.3.5.** Let us now consider several particular classes of random Gabor windows and estimate the frame bounds for the respective Gabor frames with the frame set $\Lambda = F \times \mathbb{Z}_M$ using Proposition 3.3.4.
(1) Steinhaus window. We first consider the case when the window \( g \) is chosen so that \( g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}, m \in \mathbb{Z}_M \), and \( y_m \) are independent uniformly distributed on \([0, 1)\). Then, for each \( m \in \mathbb{Z}_M \), \( M \sum_{k \in F} |g(m-k)|^2 = |F| \), and thus \( \Phi'_\Lambda \Phi'_\Lambda = |F|I_M \). That is, \((g, \Lambda)\) is a tight frame in this case.

(2) Gaussian window. For a Gaussian window \( g \sim \mathcal{CN}(0, \frac{1}{M}I_M) \), we have

\[
\sigma^2_m(\Phi'_\Lambda) = M \sum_{k \in F} |g(m-k)|^2 = \sum_{k \in F} \left( \frac{1}{2} 2Mr(m-k)^2 + \frac{1}{2} 2Ms(m-k)^2 \right),
\]

where \( r(m-k) = \Re(g(m-k)) \) denotes the real part of \( g(m-k) \), and \( s(m-k) = \Im(g(m-k)) \) denotes its imaginary part. Since, for \( k \in F \), \( \sqrt{2Mr(m-k)}, \sqrt{2Ms(m-k)} \sim \text{i.i.d.} \mathcal{N}(0, 1) \) are independent standard Gaussian random variables, we can apply Lemma 2.2.6 to obtain that, for any \( t > 0 \),

\[
\mathbb{P}\left\{ \sigma^2_m(\Phi'_\Lambda) \geq |F| + \sqrt{2|F|t} + t \right\} \leq e^{-t};
\]

\[
\mathbb{P}\left\{ \sigma^2_m(\Phi'_\Lambda) \leq |F| - \sqrt{2|F|t} \right\} \leq e^{-t}.
\]

Then, setting \( t = 2|F| \) in the first equation and \( t = \frac{1}{8}|F| \) in the second one, we obtain

\[
\mathbb{P}\left\{ \sigma^2_m(\Phi'_\Lambda) \geq 5|F| \right\} \leq e^{-2|F|};
\]

\[
\mathbb{P}\left\{ \sigma^2_m(\Phi'_\Lambda) \leq \frac{1}{2}|F| \right\} \leq e^{-\frac{|F|}{8}}.
\]

Suppose now that \( |F| \geq C \log M \), for some sufficiently large constant \( C > 0 \). Then, combining the probability estimates obtained above and taking the union bound over all \( m \in \mathbb{Z}_M \), we obtain that, with high probability,

\[
\frac{1}{2}|F| < \sigma^2_m(\Phi'_\Lambda) < 5|F|,
\]

for all \( m \in \mathbb{Z}_M \). In particular, for the frame bounds of \((g, \Lambda)\) we have

\[
\frac{1}{2}|F| < \sigma^2_{\min}(\Phi'_\Lambda) \leq \sigma^2_{\max}(\Phi'_\Lambda) < 5|F|. \tag{3.8}
\]

(3) Window, uniformly distributed on \( \mathbb{S}^{M-1} \). It is a well-known fact that a window \( g \), uniformly distributed on the unit sphere \( \mathbb{S}^{M-1} \), can be written in the form \( g = h/\|h\|_2 \), where \( h \sim \mathcal{CN}(0, \frac{1}{M}I_M) \) \[57\]. Moreover, Lemma 2.2.7 shows that, for some \( C > 0 \), \( \frac{1}{2} \leq \|h\|_2 \leq 2 \) with probability at least \( 1 - e^{-CM} \). Thus, with the same probability,

\[
\frac{1}{4} M \sum_{k \in F} |h(m-k)|^2 \leq M \sum_{k \in F} |g(m-k)|^2 \leq 4M \sum_{k \in F} |h(m-k)|^2.
\]

Combining this with (3.8), we obtain that with high probability

\[
\frac{1}{8}|F| < \sigma^2_{\min}(\Phi'_\Lambda) \leq \sigma^2_{\max}(\Phi'_\Lambda) < 20|F|.
\]
The examples above show that, in the case when $\Lambda$ has a regular structure and window $g$ is random, the Gabor frame $(g, \Lambda)$ has frame bounds that are quite close to each other, and, thus, is well-conditioned. Let us now turn into consideration the case when $\Lambda$ is a general subset of $\mathbb{Z}_M \times \mathbb{Z}_M$. We start with showing the following technical lemma, which follows the idea of [66, Lemma 3.4].

**Lemma 3.3.6.** Let $g$ be a Steinhaus window, that is, $g(j) = \frac{1}{\sqrt{M}} e^{2\pi i y_j}$, $j \in \mathbb{Z}_M$, with $y_j$ independent uniformly distributed on $[0, 1)$. Consider a Gabor system $(g, \Lambda)$ with $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$. Then, for any $m \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{P}\left\{ \left| \frac{1}{M} \right| (1 - \delta) \leq \sigma^2_{\min}(\Phi^*_{\Lambda}) \leq \sigma^2_{\max}(\Phi^*_{\Lambda}) \leq \frac{1}{M} (1 + \delta) \right\} \geq 1 - \frac{M^{2m}}{|\Lambda|^2m} \delta^{-2m} \mathbb{E}(\mathrm{Tr} H^{2m}),$$

where $H = \Phi^*_{\Lambda} \Phi^*_{\Lambda} - \frac{|\Lambda|}{M} I_M$. Furthermore, for any $m \in \mathbb{N}$,

$$\mathbb{E}(\mathrm{Tr} H^m) = \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, \ k_1, k_2 \in \Lambda} \sum_{(k_m, \ell_m) \in \Lambda} \cdots \sum_{(k_1, \ell_1) \in \Lambda} e^{\frac{2\pi i}{M} \sum_{t=1}^{m} \ell_t (j_t - j_{t+1})} E_{j_1 \ldots j_m, k_1 \ldots k_m},$$

where $E_{j_1 \ldots j_m, k_1 \ldots k_m} = \frac{1}{M^m}$, if there exists a bijection $\alpha : \{1, \ldots, m\} \to \{1, \ldots, m\}$, such that $j_t - k_t = j_{\alpha(t)} - k_{\alpha(t)-1}$, for all $t \in \{1, \ldots, m\}$; and $E_{j_1 \ldots j_m, k_1 \ldots k_m} = 0$, otherwise.

**Proof.** First, for $H = \Phi^*_{\Lambda} \Phi^*_{\Lambda} - \frac{|\Lambda|}{M} I_M$, we note that

$$\mathbb{P}\left\{ \left| \frac{1}{M} \right| (1 - \delta) \leq \sigma^2_{\min}(\Phi^*_{\Lambda}) \leq \sigma^2_{\max}(\Phi^*_{\Lambda}) \leq \frac{1}{M} (1 + \delta) \right\} = \mathbb{P}\left\{ \|H\|_2 \leq \frac{|\Lambda|}{M} \right\}. $$

Using Markov’s inequality, the fact that the Frobenius norm majorizes the operator norm, and the fact that $H$ is self-adjoint, for any $m \in \mathbb{N}$ we have

$$\mathbb{P}\left\{ \|H\|_2 > \frac{|\Lambda|}{M} \delta \right\} = \mathbb{P}\left\{ \|H\|_2 > \frac{|\Lambda|^2}{M^{2m}} \delta^{2m} \right\} \leq \frac{M^{2m}}{|\Lambda|^2m} \delta^{-2m} \mathbb{E}(\|H\|_2^m)$$

$$= \frac{M^{2m}}{|\Lambda|^2m} \delta^{-2m} \mathbb{E}(\|H^m\|_2^m) \leq \frac{M^{2m}}{|\Lambda|^2m} \delta^{-2m} \mathbb{E}(\|H^m\|_F^2)$$

$$= \frac{M^{2m}}{|\Lambda|^2m} \delta^{-2m} \mathbb{E}(\mathrm{Tr} H^{2m}). \quad (3.9)$$

That is, to conclude the desired result, we aim to estimate the trace expectation $\mathbb{E}(\mathrm{Tr} H^{2m})$. For any $j_1, j_2 \in \mathbb{Z}_M$,

$$\Phi^*_{\Lambda} \Phi^*_{\Lambda}(j_1, j_2) = \sum_{(k, \ell) \in \Lambda} e^{2\pi i (j_1 - j_2)/M} g(j_1 - k) g(j_2 - k).$$

Thus, since $|g(j)| = \frac{1}{\sqrt{M}}$, for all $j \in \mathbb{Z}_M$, for $H$ we have

$$H(j_1, j_2) = \left\{ \begin{array}{ll}
\sum_{(k, \ell) \in \Lambda} e^{2\pi i (j_1 - j_2)/M} g(j_1 - k) g(j_2 - k), & \text{if } j_1 \neq j_2; \\
0, & \text{if } j_1 = j_2.
\end{array} \right.$$
Thus, under the convention that $k \in \mathbb{Z}_M$, we recursively obtain

$$H^2(j_1, j_3) = \sum_{j_2 \in \mathbb{Z}_M} H(j_1, j_2)H(j_2, j_3)$$

$$= \sum_{j_2 \in \mathbb{Z}_M, k_1, k_2 \in \Lambda} e^{2\pi i \ell_1(j_2 - j_1) + \ell_2(j_2 - j_3)} g(j_1 - k_1)g(j_2 - k_1)g(j_2 - k_2)g(j_3 - k_2);$$

$$H^3(j_1, j_4) = \sum_{j_3 \in \mathbb{Z}_M} H^2(j_1, j_3)H(j_3, j_4)$$

$$= \sum_{j_3 \in \mathbb{Z}_M, j_2 \in \mathbb{Z}_M, k_1, k_2, k_3 \in \Lambda} e^{2\pi i \sum_{t=1}^3 \ell_t(j_t - j_{t-1})} \prod_{t=1}^3 g(j_t - k_t)g(j_{t+1} - k_t);$$

and, in general,

$$H^m(j_1, j_{m+1}) = \sum_{j_m \in \mathbb{Z}_M} H^{m-1}(j_1, j_m)H(j_m, j_{m+1})$$

$$= \sum_{j_m \in \mathbb{Z}_M, j_m \neq j_{m+1}} \cdots \sum_{j_3 \in \mathbb{Z}_M, j_2 \in \mathbb{Z}_M, k_1, k_2, k_3 \in \Lambda} \sum_{(k_3, k_4) \in \Lambda} e^{2\pi i \sum_{t=1}^m \ell_t(j_t - j_{t-1})} \prod_{t=1}^m g(j_t - k_t)g(j_{t+1} - k_t),$$

and

$$\text{Tr}(H^m) = \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, k_1, k_2, \ldots, k_m \in \Lambda} e^{2\pi i \sum_{t=1}^m \ell_t(j_t - j_{t-1})} \prod_{t=1}^m g(j_t - k_t)g(j_{t+1} - k_t),$$

and

$$\mathbb{E}(\text{Tr}(H^m)) = \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, k_1, k_2, \ldots, k_m \in \Lambda} \sum_{(k_3, k_4) \in \Lambda} e^{2\pi i \sum_{t=1}^m \ell_t(j_t - j_{t-1})} E_{j_1, \ldots, j_m, k_1, \ldots, k_m},$$

where $E_{j_1, \ldots, j_m, k_1, \ldots, k_m} = \mathbb{E}\left(\prod_{t=1}^m g(j_t - k_t)g(j_{t+1} - k_t)\right)$.

Let us compute $E_{j_1, \ldots, j_m, k_1, \ldots, k_m}$ now. Since $g(j)$, $j \in \mathbb{Z}_M$, are independent, the expectation can be factored into a product of the form

$$\mathbb{E}\left(\prod_{t=1}^m g(j_t - k_t)g(j_{t+1} - k_t)\right) = \prod_{j \in \mathbb{Z}_M} \mathbb{E}\left(g(j)^{\mu_j}g(j)^{\nu_j}\right),$$

for some $\mu_j, \nu_j \in \mathbb{N} \cup \{0\}$. Moreover, since $\sqrt{M}g(j)$ is uniformly distributed on the unit torus $\{z \in \mathbb{C} : \|z\|_2 = 1\}$ and $\mathbb{E}(g(j)) = 0$, we have

$$\mathbb{E}\left(g(j)^{\mu_j}g(j)^{\nu_j}\right) = \begin{cases} \mathbb{E}(|g(j)|^{2\mu_j}) = \frac{1}{M^{\mu_j}}, & \mu_j = \nu_j; \\ 0, & \mu_j \neq \nu_j. \end{cases}$$

Thus, under the convention that $k_0 = k_m$,

$$E_{j_1, \ldots, j_m, k_1, \ldots, k_m} = \begin{cases} \frac{1}{M^{\mu_j}}, & \text{if } \exists \text{ bijection } \alpha : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}, \\
\quad \text{s.t. } \forall t \in \{1, \ldots, m\} \ j_t - k_t = j_{\alpha(t)} - k_{\alpha(t)-1}; \\
0, & \text{otherwise.} \end{cases}$$

This concludes the proof. \qed
Using Lemma 3.3.6 with \( m = 1 \), we obtain the following bound on \( \sigma^2_{\text{max}}(\Phi_A^*) \) for a general \( \Lambda \), which does not depend on the structure of \( \Lambda \) but only on its cardinality.

**Theorem 3.3.7.** Let \( g \) be a Steinhaus window, that is, \( g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}, \ m \in \mathbb{Z}_M \), with \( y_m \) independent uniformly distributed on \([0, 1)\). Consider a Gabor system \((g, \Lambda)\) with \( \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M \). Then, for each fixed \( \varepsilon \in (0, 1) \), with probability at least \( 1 - \varepsilon \),

\[
\sigma^2_{\text{max}}(\Phi_A^*) \leq \frac{|A|}{M} + \sqrt{\frac{|A|}{\varepsilon} \left(1 - \frac{|A|}{M^2}\right)}.
\]

**Proof.** Applying Lemma 3.3.6 with \( m = 1 \), we obtain that, for \( H = \Phi_A \Phi_A^* - \frac{|A|}{M} I_M \) and any \( \delta > 0 \),

\[
P \left\{ \sigma^2_{\text{max}}(\Phi_A^*) > \frac{|A|}{M} (1 + \delta) \right\} \leq \frac{M^2}{|A|^2} \delta^{-2}\mathbb{E}(\text{Tr} \ H^2).
\]

And, moreover, we have

\[
\mathbb{E}(\text{Tr} \ H^2) = \sum_{j_1, j_2 \in \mathbb{Z}_M} \sum_{k_1, k_2 \in \mathbb{Z}_M} \sum_{(k_1, k_2) \in \Lambda} \sum_{(j_1, j_2) \in \Lambda} e^{2\pi i (j_1 - j_2) (k_1 - k_2)} E_{j_1 j_2, k_1 k_2}
\]

where \( E_{j_1 j_2, k_1 k_2} = \frac{1}{M^2} \), if there exists a bijection \( \alpha : \{1, 2\} \to \{1, 2\} \), such that, for every \( t \in \{1, 2\} \), \( j_t - k_t = j_{\alpha(t)} - k_{\alpha(t)} - 1 \); and \( E_{j_1 j_2, k_1 k_2} = 0 \), otherwise. If we have \( j_1 - k_1 = j_2 - k_2 \), or \( j_2 - k_2 = j_1 - k_1 \), it follows that \( j_1 = j_2 \), which is a contradiction. Thus we have

\[
\begin{cases} j_1 - k_1 = j_2 - k_2 \\ j_2 - k_2 = j_1 - k_1 \end{cases} \iff k_1 = k_2.
\]

For each \( k \in \mathbb{Z}_M \), let us consider the set \( A_k = \{ \ell \in \mathbb{Z}_M, (k, \ell) \in \Lambda \} \). Clearly, \( \sum_{k \in \mathbb{Z}_M} |A_k| = |A| \). Then, for the expectation of \( \text{Tr} \ H^2 \), we have the following.

\[
\mathbb{E}(\text{Tr} \ H^2) = \frac{1}{M^2} \sum_{j_1, j_2 \in \mathbb{Z}_M, \ j_1 \neq j_2} \sum_{k \in \mathbb{Z}_M} \sum_{\ell_1 \in A_k} \sum_{\ell_2 \in A_k} e^{2\pi i (\ell_1 - \ell_2) (j_1 - j_2)}
\]

\[
= \frac{1}{M^2} \sum_{j_1 \in \mathbb{Z}_M} \sum_{k \in \mathbb{Z}_M} \sum_{\ell_1 \in A_k} \left( \sum_{j_2 \in \mathbb{Z}_M} \sum_{\ell_2 \in A_k, \ \ell_2 \neq \ell_1} e^{2\pi i (\ell_1 - \ell_2) (j_1 - j_2)} + \sum_{j_2 \in \mathbb{Z}_M, \ j_2 \neq j_1} 1 \right)
\]

\[
= \frac{1}{M^2} \sum_{j_1 \in \mathbb{Z}_M} \sum_{k \in \mathbb{Z}_M} \sum_{\ell_1 \in A_k} \left( \sum_{\ell_2 \in A_k, \ \ell_2 \neq \ell_1} (-1) + M - 1 \right)
\]

\[
= \frac{1}{M^2} \sum_{k \in \mathbb{Z}_M} \sum_{\ell_1 \in A_k} \left( \sum_{\ell_2 \in A_k, \ \ell_2 \neq \ell_1} (|A_k| - 1)(-1) + M - 1 \right)
\]

\[
= \frac{1}{M^2} \sum_{k \in \mathbb{Z}_M} (|A_k| - 1)(-1) + M - 1 = \frac{1}{M} \sum_{k \in \mathbb{Z}_M} |A_k| (M - |A_k|)
\]

\[
= \sum_{k \in \mathbb{Z}_M} |A_k| - \frac{1}{M} \sum_{k \in \mathbb{Z}_M} |A_k|^2 \leq |A| \left(1 - \frac{|A|}{M^2}\right).
\]

The last step here is due to the fact that \( \sum_{k \in \mathbb{Z}_M} |A_k|^2 \geq \frac{1}{M^2} \left( \sum_{k \in \mathbb{Z}_M} |A_k| \right)^2 \).
Then, setting \( \delta = \sqrt{\frac{M^2 - |\Lambda|}{\varepsilon |\Lambda|}} \) for some \( \varepsilon \in (0,1) \), we obtain

\[
\mathbb{P}\left\{ \sigma_{\text{max}}^2(\Phi_\Lambda^*) > \frac{|\Lambda|}{M} + \sqrt{\frac{|\Lambda|}{\varepsilon} \left(1 - \frac{|\Lambda|}{M^2}\right)} \right\} \leq \varepsilon,
\]

which concludes the proof. \( \square \)

We note that the bound obtained in Theorem 3.3.7 is tight for a full Gabor frame, when \( \Lambda = \mathbb{Z}_M \times \mathbb{Z}_M \). In the case when \( |\Lambda| = \alpha M^2 \), the proven bound gives \( \sigma_{\text{max}}^2(\Phi_\Lambda^*) \leq \left( \alpha + \sqrt{\frac{\alpha(1-\alpha)}{\varepsilon}} \right) M \) with probability at least \( 1 - \varepsilon \).

In the following theorem, we consider the case of a randomly selected set \( \Lambda \). Roughly speaking, this result shows that, for any \( \varepsilon \in (0,1) \), a generic subframe \((g, \Lambda)\) of \((g, \mathbb{Z}_M \times \mathbb{Z}_M)\) with \( |\Lambda| = O(M^{1+\varepsilon}) \log M \) is well-conditioned with high probability.

**Theorem 3.3.8.** Let \( g \) be a Steinhaus window, that is, \( g(j) = \frac{1}{\sqrt{M}} e^{2\pi i y_j} \), \( j \in \mathbb{Z}_M \), with \( y_j \) independent uniformly distributed on \([0,1)\). For any fixed even \( m \in \mathbb{N} \), consider a Gabor system \((g, \Lambda)\) with a random set \( \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M \) constructed so that events \( \{ (k, \ell) \in \Lambda \} \) are independent for all \((k, \ell) \in \mathbb{Z}_M \times \mathbb{Z}_M \) and have probability \( \tau = \frac{C \log M}{M^{1+\varepsilon}} \), where \( C > 0 \) is a sufficiently large constant depending only on \( m \). Then, with high probability (with respect to the choice of \( \Lambda \)),

\[
\mathbb{P}\left\{ \frac{|\Lambda|}{M} (1-\delta) \leq \sigma_{\text{min}}^2(\Phi_\Lambda^*) \leq \sigma_{\text{max}}^2(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M} (1+\delta) \right\} \geq 1 - \varepsilon,
\]

where \( \varepsilon \in (0,1) \) depends on \( m, \delta \), and the choice of \( C \).

**Proof.** Let us set \( H = \Phi_\Lambda \Phi_\Lambda^* - \frac{|\Lambda|}{M} I_M \). Applying Lemma 3.3.6, we obtain that, for every \( m \in \mathbb{N} \) and \( \delta > 0 \),

\[
\mathbb{P}\left\{ \frac{|\Lambda|}{M} (1-\delta) \leq \sigma_{\text{min}}^2(\Phi_\Lambda^*) \leq \sigma_{\text{max}}^2(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M} (1+\delta) \right\} \geq 1 - \frac{M^m}{|\Lambda|^m} \delta^m \mathbb{E}(\text{Tr} H^m),
\]

where \( \mathbb{E}(\text{Tr} H^m) = \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M} \sum_{(k_1, \ell_1)} \cdots \sum_{(k_m, \ell_m)} e^{2\pi i \sum_{t=1}^m \xi_t (j_t - j_{t+1})} E_{j_1 \ldots j_m}^{k_1 \ldots k_m} \)

where \( E_{j_1 \ldots j_m}^{k_1 \ldots k_m} = \frac{1}{M^m} \) if there exists a permutation \( \alpha \in \Sigma_m \), such that, for every \( t \in \{1, \ldots, m\} \), \( j_t - k_t = j_{\alpha(t)} - k_{\alpha(t)-1} \); and \( E_{j_1 \ldots j_m}^{k_1 \ldots k_m} = 0 \) otherwise. Here, \( \Sigma_m \) denotes the group of permutations of \( \{1, \ldots, m\} \). That is, \( \alpha \in \Sigma_m \) is a bijection \( \alpha : \{1, \ldots, m\} \to \{1, \ldots, m\} \).

For \( k \in \mathbb{Z}_M \), let us denote by \( A_k \) the set \( A_k = \{ \ell \in \mathbb{Z}_M, \text{ s.t. } (k, \ell) \in \Lambda \} \). After rearranging the sum in the trace formula above, we have

\[
\mathbb{E}(\text{Tr} H^m) = \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M} \sum_{k_1, k_2, \ldots, k_m \in \mathbb{Z}_M} E_{j_1 \ldots j_m}^{k_1 \ldots k_m} \prod_{t=1}^m \ell_t \sum_{\ell \in A_{k_t}} e^{2\pi i \sum_{t=1}^m \xi_t (j_t - j_{t+1})}
\]

for every \( \xi \in \mathbb{C} \).

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We note that, by the construction of $\Lambda$, each set $A_{k_t}$, $t \in \{1, \ldots, m\}$, is a random subset of $\mathbb{Z}_M$, such that the events $\{ \ell \in A_{k_t} \}$, $\ell \in \mathbb{Z}_M$, are independent and have probability $\tau$. Then Corollary 2.2.10 implies that, for every $t \in \{1, \ldots, m\}$ and a constant $C' > 4\sqrt{2}$,

$$\mathbb{P} \left\{ \max_{q \in \mathbb{Z}_M, q \neq 0} \left| \sum_{\ell \in A_{k_t}} e^{2\pi i \ell q/M} \right| < C' \log M \right\} \geq 1 - \frac{1}{M^{\frac{C'}{2\tau^2}}},$$

In particular, $\max_{j_t, j_{t+1} \in \mathbb{Z}_M, j_t \neq j_{t+1}} \left| \sum_{\ell_t \in A_{k_t}} e^{2\pi i \ell_t (j_t - j_{t+1})} \right| < C' \log M$, with probability at least $1 - \frac{1}{M^{\frac{C'}{2\tau^2}}}$, By taking the union bound over all $t \in \{1, \ldots, m\}$, we conclude that, with probability at least $1 - \frac{m}{M^{\frac{C'}{2\tau^2}}}$,

$$\left| \prod_{t=1}^m \sum_{\ell_t \in A_{k_t}} e^{2\pi i \ell_t (j_t - j_{t+1})} \right| < C'^m \log^m M.$$

Then, applying the triangular inequality to the trace formula, we obtain that, on an event $X$ of probability at least $1 - \frac{m}{M^{\frac{C'}{2\tau^2}}}$,

$$\mathbb{E} (\text{Tr} H^m) \leq \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, k_1, k_2, \ldots, k_m \in \mathbb{Z}_M} E_{j_1 \ldots j_m} \sum_{k_1 \ldots k_m} \left| \prod_{t=1}^m \sum_{\ell_t \in A_{k_t}} e^{2\pi i \ell_t (j_t - j_{t+1})} \right| < C'^m \log^m M \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, k_1, k_2, \ldots, k_m \in \mathbb{Z}_M} E_{j_1 \ldots j_m} \sum_{k_1 \ldots k_m} \left| \prod_{t=1}^m \sum_{\ell_t \in A_{k_t}} e^{2\pi i \ell_t (j_t - j_{t+1})} \right|.$$

A permutation $\alpha \in \Sigma_m$ can be presented as a product

$$\alpha = (i_{11} i_{12} \ldots i_{1r_1})(i_{21} i_{22} \ldots i_{2r_2}) \ldots (i_{s1} i_{s2} \ldots i_{sr_s}) \quad (3.10)$$

of disjoint cycles, where $r_1 + r_2 + \cdots + r_s = m$, and, for each $p \in \{1, \ldots, s\}$, $\alpha(i_{pq}) = i_{p(q+1)}$ for $q \in \{1, \ldots, r_p - 1\}$ and $\alpha(i_{pr_p}) = i_{p1}$.

Suppose that we have $k_1, \ldots, k_m$ fixed. Then $E_{j_1 \ldots j_m} \neq 0$ if and only if there exists $\alpha \in \Sigma_m$, such that $j_t - j_{\alpha(t)} = k_t - k_{\alpha(t)} - 1$, for all $t \in \{1, \ldots, m\}$. Assuming that $\alpha$ has $s$ cycles in the disjoint cycle decomposition (3.10), this condition can be rewritten in the form of $s$ systems of linear equations for $j_1, \ldots, j_m$. Namely, for each $p \in \{1, \ldots, s\}$, we have

$$j_{ip_1} - j_{ip_2} = k_{ip_1} - k_{ip_2} - 1,$$

$$j_{ip_2} - j_{ip_3} = k_{ip_2} - k_{ip_3} - 1,$$

$$\ldots$$

$$j_{ip_{r_p}} - j_{ip_1} = k_{ip_{r_p}} - k_{ip_1} - 1.$$

Note that the system (3.11) has rank $r_p - 1$. Furthermore, summing up all the equations, on the left hand side we obtain zero. So, (3.11) has $M$ different solutions if

$$\sum_{q=1}^{r_p} k_{ip_q} = \sum_{q=1}^{r_p} k_{ip_{q-1}}. \quad (3.12)$$
and does not have a solution otherwise. Moreover, if $s \neq 1$, that is, $r_p < m$, then the sets of indices $\{i_{pq}\}_{q=1}^{r_p}$ on the left hand side of (3.12) and $\{i_{pq} - 1\}_{q=1}^{r_p}$ on the right hand side of (3.12) are different. Indeed, suppose that $\{i_{pq}\}_{q=1}^{r_p} = \{i_{pq} - 1\}_{q=1}^{r_p}$, and let $i_{pq0} = \min_{q\in\{1,...,r_p\}} i_{pq}$ be the smallest element in this set. Since $i_{pq0} - 1$ is also an element of $\{i_{pq}\}_{q=1}^{r_p}$, we have $i_{pq0} - 1 \geq i_{pq0}$, which implies $i_{pq0} = 1$ and $i_{pq0} - 1 = m$. Then, since $m \in \{i_{pq}\}_{q=1}^{r_p}$, we also have $m - 1 \in \{i_{pq}\}_{q=1}^{r_p}$. Proceeding the argument by induction, we obtain $\{i_{pq}\}_{q=1}^{r_p} = \{1, \ldots, m\}$, which is a contradiction. Without loss of generality, we can assume that $i_{pr_p} \notin \{i_{pq} - 1\}_{q=1}^{r_p}$, for every $p \in \{1, \ldots, s\}$.

It follows that, for each cycle in the cycle decomposition (3.10), except the last one, equation (3.12) is a nontrivial linear relation for $k_t$, $t \in \{1, \ldots, m\}$. For the last cycle the relation follows automatically, assuming (3.12) is satisfied for each $p \in \{1, \ldots, s - 1\}$. So, for the system of linear equations for $j_1, \ldots, j_m$ to have a solution, $k_{i_{pr_p}}$, $p \in \{1, \ldots, s - 1\}$, should be determined by $\{k_1, \ldots, k_m\} \setminus \{k_{i_{pr_p}}\}_{p=1}^{s-1}$ using equations (3.12). It this case the number of different solutions is $M^s$.

Then, for the expectation of the trace of $H^m$, on the event $X$ we have

$$
\mathbb{E} \left( \text{Tr} H^m \right) \leq C^m \log^m M \sum_{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M} \sum_{k_1, k_2, \ldots, k_m \in \mathbb{Z}_M} E_{j_1 \ldots j_m}^1 \sum_{k_1 \ldots k_m} 1_{M^m}^{-1} 
\leq C^m \log^m M \sum_{s=1}^{m} S(m, s) \sum_{j_1, \ldots, j_s \in \mathbb{Z}_M} \sum_{k_1, \ldots, k_{i_{(r_s - 1)}}, \ldots, k_{s-1}, \ldots, k_s} 1_{M^m}^{-1} 
= C^m \log^m M \sum_{s=1}^{m} S(m, s) M^s M^{m-s+1} 
= C^m \log^m M \sum_{s=1}^{m} S(m, s) = C^m m! M \log^m M,
$$

where $S(m, s)$ denotes the Stirling number of the first kind, equal to the number of permutations in $\Sigma_m$ with exactly $s$ cycles in the disjoint cycle decomposition.

Moreover, the cardinality of $\Lambda$ is given by a sum of $M^2$ independent Bernoulli random variables with success probability $\tau = \frac{C \log M}{\log^m M}$. More precisely,

$$
|\Lambda| = \sum_{(k, \ell) \in \mathbb{Z}_M \times \mathbb{Z}_M} 1_{\Lambda}(k, \ell).
$$

Then Hoeffding’s inequality (Lemma 2.2.5) applied with $t = \frac{C \log M}{2 M \log^2 M}$ implies

$$
\mathbb{P} \left\{ |\Lambda| \leq \frac{1}{2} C M^{1+\frac{2}{\log^2 M}} \log M \right\} \leq e^{-2C^2 M^{2/\log^2 M} \log^2 M}.
$$

That is, $|\Lambda| > \frac{1}{2} C M^{1+\frac{2}{\log^2 M}} \log M$ on an event $Y$ of probability at least $1 - e^{-2C^2 M^{2/\log^2 M} \log^2 M}$.

Then, on the event $X \cap Y$, which has probability at least $1 - \frac{\tilde{C}}{M^{2/\log^2 M}}$, for some $\tilde{C} > 0$, the obtained estimates for the trace expectation and frame set cardinality
lead to the following probability bound for the singular values estimates.

\[
\begin{align*}
\mathbb{P}\left\{ \frac{|\Lambda|}{M} (1 - \delta) \leq \sigma_{\min}^2(\Phi_{\Lambda}^*) \leq \sigma_{\max}^2(\Phi_{\Lambda}^*) \leq \frac{|\Lambda|}{M} (1 + \delta) \right\} & \geq 1 - \frac{M^m}{|\Lambda|^m} \delta^{-m} \mathbb{E}(\text{Tr} H^m) \\
& \geq 1 - C^m m! \delta^{-m} \frac{1}{2^m} C^m M^{m+1} \log^m M \log^2 M = 1 - \left( \frac{2C'}{C} \right)^m m! \delta^{-m}.
\end{align*}
\]

This concludes the proof provided \( C \) is chosen to be large enough.

\( \square \)

### 3.3.1 Numerical results

In this section we further investigate singular values of Gabor frames with a random window using numerical simulations. In particular, we aim to numerically analyse the bounds on the extreme singular values of the analysis matrix \( \Phi_{\Lambda}^* \) of a Gabor frame \( (g, \Lambda) \) with a Steinhaus window \( g \) in the case when \( \Lambda \) is a random subset of \( \mathbb{Z}_M \times \mathbb{Z}_M \).

Let us fix an even \( m \in \mathbb{N} \), and let \( C > 0 \) be a sufficiently large constant depending on \( m \). Consider a random \( \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M \), such that the events \( \{(k, \ell) \in \Lambda\} \) are independent for all \( (k, \ell) \in \mathbb{Z}_M \times \mathbb{Z}_M \) and have probability \( \tau = \frac{C \log M}{M^{2m}} \). Theorem 3.3.8 ensures that, with high probability (with respect to the choice of \( \Lambda \)),

\[ \mathbb{P}\left\{ \frac{|\Lambda|}{M} (1 - \delta) \leq \sigma_{\min}^2(\Phi_{\Lambda}^*) \leq \sigma_{\max}^2(\Phi_{\Lambda}^*) \leq \frac{|\Lambda|}{M} (1 + \delta) \right\} \geq 1 - \varepsilon, \]

where \( \varepsilon \in (0, 1) \) depends on \( m, \delta, \) and the choice of \( C \). To illustrate Theorem 3.3.8, we use two sets of numerical simulations.

In the first set of numerical simulations, we investigate the behavior of the singular values of the analysis matrix \( \Phi_{\Lambda}^* \) of a Gabor frame \( (g, \Lambda) \) with a Steinhaus window \( g \) and set \( \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M \) selected at random, so that \( |\Lambda| = O(M) \) with high probability. The obtained numerical results suggest that, in the case when random \( \Lambda \) is constructed as described in Theorem 3.3.8 with \( \tau = \frac{C}{M} \), there exist constants \( 0 < k < K \) not depending on the ambient dimension \( M \), such that all the singular values of the analysis matrix \( \Phi_{\Lambda}^* \) are inside the interval \( \left[k \frac{|\Lambda|}{M}, K \frac{|\Lambda|}{M}\right] \) with high probability, see Figure 3.3 (left). The right hand side of Figure 3.3 shows the distribution of the singular values of \( \Phi_{\Lambda}^* \) over this interval for the selected dimensions \( M = 100, 150, 200, 250, 300 \).

We use the second set of simulations to investigate the behavior of the trace of the matrix \( H = \Phi_{\Lambda} \Phi_{\Lambda}^* - \frac{|\Lambda|}{M} I_M \), where \( \Phi_{\Lambda} \) is the synthesis matrix of a Gabor frame \( (g, \Lambda) \) with a Steinhaus window \( g \). It follows from Lemma 3.3.6 that

\[ \mathbb{P}\left\{ \sigma_{\min}^2(\Phi_{\Lambda}^*) \leq \frac{|\Lambda|}{M} (1 - \delta) \text{ or } \sigma_{\max}^2(\Phi_{\Lambda}^*) \geq \frac{|\Lambda|}{M} (1 + \delta) \right\} \leq \frac{M^{2m}}{|\Lambda|^m} \delta^{-2m} \mathbb{E}(\text{Tr} H^{2m}). \]

In other words, the normalized trace expectation \( \frac{M^m}{|\Lambda|^m} \mathbb{E}(\text{Tr} H^m) \) is used to estimate the probability of the “failure” event when either the minimal singular value of the analysis matrix \( \Phi_{\Lambda}^* \) is too small or the maximal singular value is too large, so that \( (g, \Lambda) \) is not well-conditioned.

For the normalized trace expectation, we consider two different constructions of \( \Lambda \), providing the average and the worst case estimates, respectively. The left
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Figure 3.3: The left hand side of the figure shows the dependence of the extreme singular values of the analysis matrix $\Phi^*_\Lambda$ of a Gabor frame $(g, \Lambda)$ on the ambient dimension $M$; and the right hand side of the figure shows the distribution of the singular values of $\Phi^*_\Lambda$ for the dimensions $M = 100, 150, 200, 250, 300$. Here, $g$ is a Steinhaus window and $\Lambda$ is chosen at random as described in Theorem 3.3.8, with $\tau = \frac{C}{M}$, that is, $|\Lambda| = O(M)$ with high probability. The number of the numerical experiments considered here is 1000. These numerical results suggest that, with high probability, the singular values of $\Phi^*_\Lambda$ lie inside an interval $[k\frac{|\Lambda|}{M}, K\frac{|\Lambda|}{M}]$, for some constants $0 < k < K$ that do not depend on $M$. This allows us to conjecture that a version of Theorem 3.3.8 is true also for $\Lambda$ with $|\Lambda| = O(M)$. In other words, the additional factor of $M^\epsilon \log M$ in the cardinality of $\Lambda$ is a side effect of the method used to prove the theorem.

Hand side of Figure 3.4 shows the numerical results in the case when $\Lambda$ is chosen at random as described in Theorem 3.3.8 with $\tau = \frac{C}{M}$. The right hand side of Figure 3.4 illustrates the case when $\Lambda$ is of the form $\Lambda = F \times \{0, 1, \ldots, \lfloor \frac{M}{2} \rfloor \}$, $F \subset \mathbb{Z}_M$. The plots show the dependence of the normalized trace expectation on the ambient dimension $M$ and the parameter $C$ in the definition of $\tau$, for a fixed $m$. The obtained numerical results suggest that, in both cases, the normalized trace expectation decreases rapidly with the dimension. This allows us to conjecture that the probability bound obtained in Theorem 3.3.8 can be further improved. Moreover, Figure 3.4 (left) shows that, in the case of randomly selected $\Lambda$, the normalized trace expectation does not seem to depend on the parameter $C$.

These numerical findings, illustrated on Figures 3.3 and 3.4, motivate the following conjecture.

**Conjecture 3.3.9.** Let $g$ be a Steinhaus window, that is, $g(j) = \frac{1}{\sqrt{M}} e^{2\pi i y_j j}$, $j \in \mathbb{Z}_M$, with $y_j$ independent uniformly distributed on $[0, 1)$. Consider a Gabor system $(g, \Lambda)$ with a random set $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ constructed so that events $\{(k, \ell) \in \Lambda\}$ are independent for all $(k, \ell) \in \mathbb{Z}_M \times \mathbb{Z}_M$ and have probability $\tau = \frac{C}{M}$, where $C > 0$ is a sufficiently large numerical constant. Then, with high probability (with respect to the choice of $\Lambda$),

$$
\mathbb{P} \left\{ \frac{|\Lambda|}{M} (1 - \delta) \leq \sigma^2_{\min}(\Phi_{\Lambda}^*) \leq \sigma^2_{\max}(\Phi_{\Lambda}^*) \leq \frac{|\Lambda|}{M} (1 + \delta) \right\} \geq 1 - \frac{c}{M},
$$

where the constant $c > 0$ depends only on $\delta$.

In other words, the additional factor of $M^{\frac{1}{\epsilon}} \log M$ in the cardinality of $\Lambda$ is a
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Figure 3.4: The figure illustrates the behavior of the numerically estimated normalized trace expectation $\frac{M}{|\Lambda|^m} E \left( \text{Tr} \left( \Phi_A \Phi_A^* - \frac{|\Lambda|}{M} I_M \right)^m \right)$, where $\Phi_A$ is the synthesis matrix of a Gabor frame $(g, \Lambda)$ with a Steinhaus window $g$. The left hand side of the figure illustrates the numerical results in the case when $\Lambda$ is chosen at random, as described in Theorem 3.3.8, with $\tau = \frac{C}{M}$; and the right hand side of the figure illustrates the case when $\Lambda = F \times \{0, 1, \ldots, \lfloor \frac{M}{2} \rfloor \}$. The plots show the dependence of the normalized trace expectation on the ambient dimension $M$ (horizontal axis) and the parameter $C$ (vertical axis), for a fixed $m$. These numerical results allow us to conjecture that the probability bound obtained in Theorem 3.3.8 can be further improved.

Erasure-robust frames

In some areas of signal processing related to communication systems or phase retrieval (see Section 4.5.2), the available frame coefficients of an (unknown) signal of interest are not only corrupted by additive noise, but some of them might be missing or be too unreliable to be used for reconstruction. In this case, the measurement frame should allow for robust signal reconstruction from incomplete set of noisy frame coefficients. Fickus and Mixon introduced the notion of numerically erasure-robust frames that formalizes this property [28].

Definition 3.3.10. For a fixed $p \in [0, 1]$ and $C \geq 1$, a frame $\Phi = \{\varphi_j\}_{j=1}^N$ is called a $(p, C)$-numerically erasure-robust frame if, for every $J \subset \{1, \ldots, N\}$ of cardinality $|J| = (1 - p)N$, the condition number of the analysis matrix of the corresponding subframe $\Phi_J = \{\varphi_j\}_{j \in J}$ satisfies $\text{Cond}(\Phi_J^*) \leq C$.

The following result shows that a Gaussian frame with independent frame vectors is a numerically erasure-robust frame [28].

Theorem 3.3.11. Fix $\varepsilon > 0$ and consider a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ such that $\varphi_j(m) \sim i.i.d. \mathcal{CN}(0,1)$, for $j \in \{1, \ldots, N\}$ and $m \in \mathbb{Z}_M$. Then $\Phi$ is a $(p, C)$-numerically erasure-robust frame with overwhelming probability provided $p$ and $C$ satisfy

$$\sqrt{\frac{M}{N}} \leq \frac{C - 1}{C + 1} \sqrt{1 - p} - \sqrt{\varepsilon} + 2p(1 - \log(p)).$$
In this section, we aim to numerically investigate robustness to erasures of Gabor frames \((g, \Lambda)\) with a random window \(g\).

We note that, since a full Gabor frame is tight, for any \(\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M\) and \(g \in S^{M-1}\),

\[
\sigma_{\max}^2 (\Phi^*_\Lambda) = \max_{x \in S^{M-1}} \sum_{\lambda \in \Lambda} |\langle x, \pi(\lambda)g \rangle|^2 \leq \max_{x \in S^{M-1}} \sum_{\lambda \in \mathbb{Z}_M \times \mathbb{Z}_M} |\langle x, \pi(\lambda)g \rangle|^2 = M.
\]

Thus we concentrate on the uniform bound on the minimal singular value \(\sigma_{\min}^2 (\Phi^*_\Lambda')\), for all subframes \((g, \Lambda')\) of \((g, \Lambda)\) with \(|\Lambda'| \geq (1-p)|\Lambda|\), where \(p\) is some fixed parameter.

For a Gabor frame \((g, \Lambda)\) with a window \(g\) uniformly distributed on the unit sphere \(S^{M-1}\), let us consider the parameter \(\Delta(p)\), \(p \in [0, 1]\), given by

\[
\Delta(p) = \min_{\Lambda', \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M} \sigma_{\min}^2 (\Phi^*_{\Lambda'}) \quad |\Lambda'| \geq (1-p)|\Lambda|.
\]

Numerical results illustrating the dependence of the value \(\Delta(1/3)\) on the dimension \(M\) are presented on Figure 3.5. They suggest that \(\Delta(p)\) is bounded away from zero by a numerical constant not depending on \(M\). More precisely, we formulate the following conjecture.

**Conjecture 3.3.12.** Consider a Gabor frame \((g, \Lambda)\) with \(g\) uniformly distributed on \(S^{M-1}\) and \(\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M\), such that \(|\Lambda| = O(M \log^\alpha M)\) (where the parameter \(\alpha \geq 0\) has to be specified). Then, for \(p \in (0, 1)\), \(\Delta(p) \geq C\) with high probability, where \(C > 0\) depends only on \(p\).
Chapter 4

Phase retrieval for Gabor frames

In this chapter we discuss the problem of phase retrieval, a non-convex inverse problem arising in many areas of imaging science and signal processing. More precisely, it is a problem of signal reconstruction from intensity measurements with respect to a measurement frame. One of the main research challenges in phase retrieval is to construct efficient recovery algorithms for specific choices of the measurement frame. While substantial progress has been achieved on this problem in the case of random measurement frames with independent frame vectors, the case of structured frames arising naturally in practical applications remains significantly less studied.

In this chapter we mainly focus on the investigation of the phase retrieval problem with time-frequency structured measurement frames. Phase retrieval with such frames arise naturally in various imaging and signal processing problems including diffraction imaging [11], speech recognition [4], radars [64, 44], and semi-blind channel estimation [78]. We design an efficient and robust recovery algorithm with a close to optimal number of time-frequency structured measurements. Robustness and stability analysis of a phase retrieval algorithm is closely linked with investigation of the geometric properties of measurement frames discussed in Chapter 3.

4.1 Problem statement and motivation

The phase retrieval problem arises naturally in many applications within a variety of fields in science and engineering. Among these applications are optics [60, 79], astronomical imaging [30], and microscopy [59]. Other fields where phase retrieval plays an important role are quantum mechanics [23] and differential geometry [8]. Even though this problem has been known for a long time, in recent years phase retrieval experiences increasing interest and has become an actively investigated research area again. This is inspired on one hand by the desire to image and investigate various kinds of individual nano-particles and on the other hand by rapid development of new imaging technologies.

As an example, let us consider the diffraction imaging problem [11]. To investigate the structure of a small particle, such as a DNA molecule, we illuminate the particle with X-rays and then measure the radiation scattered from it. When X-ray waves pass by an object and are measured in the far field, detectors are not able to capture the phase of the waves reaching them, but only their magnitudes. The measurements obtained in this way are of the form of pointwise squared absolute values of the Fourier transform of the object $x$, that is, the measurement map $A$.
is given by $A(x) = \{|\mathcal{F}(x)(n)|^2\}_{n \in \Omega}$ where $\Omega$ is the sampling grid. Since $A$ is not injective, some additional a priori information on the object $x$ is needed for reconstruction. For instance, knowledge of the chemical interactions between parts of a DNA molecule were used for the construction of the DNA double helix model in the Nobel Prize winning work of Watson, Crick, and Wilkins [80].

One way to overcome this non-injectivity when no a priori information is available is masking. To modify the phase front, one can insert a known mask after the object, as shown on Figure 4.1. The measurement map in this case is given by $A(x) = \{|\mathcal{F}(x \odot f_t)(n)|^2\}_{n \in \Omega, t \in I}$, where $f_t, t \in I$, are the masks used, and $\odot$ denotes pointwise multiplication. By increasing the number of measurements in this way, we can reduce the ambiguity in the reconstruction of signal $x$.

Since the problem of signal reconstruction from the magnitudes of its Fourier coefficients is particularly hard to handle, a more general frame theoretical setting is frequently considered. Namely, for $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ being a frame, that is, a possibly overcomplete spanning set of $\mathbb{C}^M$, we aim to recover $x \in \mathbb{C}^M$ from its phaseless squared frame coefficients $A_\Phi(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$.

Note that the masked Fourier coefficients of the signal $x$ with masks $\{f_t\}_{t \in I} \subset \mathbb{C}^M$ can be viewed as the frame coefficients of $x$ with respect to the frame $\Phi$ given by $\Phi = \{\varphi_{t,j}\}$, where $\varphi_{t,j}(m) = \frac{e^{2\pi i jm/M}}{\sqrt{M}} f_t(m)$.

Clearly, the measurement map $A_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ defined above is never injective. Even in optimal setting, $x$ can be reconstructed from intensity measurements only up to a global phase. Indeed, for every real $\theta$, the signals $x$ and $e^{i\theta}x$ produce the same intensity measurements. Thus, the goal of phase retrieval is to reconstruct the equivalence class $[x] \in \mathbb{C}^M/\sim$ of $x$, where $x \sim y$ if and only if $x = e^{i\theta}y$ for some $\theta \in [0, 2\pi)$. In the sequel, we are going to identify $x$ with its equivalence class $[x] \in \mathbb{C}^M/\sim$ and discuss injectivity of $A_\Phi$ as a map defined on $\mathbb{C}^M/\sim$.

Obviously, not every frame gives rise to an injective measurement map. But even in the case when $A_\Phi$ is known to be injective, the problem of reconstructing $[x]$ from $A_\Phi(x)$ is NP-hard in general [71]. So, the main goals in this area of applied mathematics is to find conditions on the number of measurements $N$ and vectors $\varphi_j$ for which there exist an efficient and robust numerical recovery algorithm.
4.2 Overview of state of the art in phase retrieval

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame, and let $x \in \mathbb{C}^M/\sim$ be a signal that we wish to reconstruct from phaseless measurements $b = A_\Phi(x)$, where the measurement map $A_\Phi : \mathbb{C}^M/\sim \to \mathbb{R}^N$ is given by

$$A_\Phi(x) = \{ |\langle x, \varphi_j \rangle|^2 \}_{j=1}^N.$$  \hfill (4.1)

There are two main directions of research on the phase retrieval problem which can be formulated as follows.

(i) For which measurement frames $\Phi$ is the map $A_\Phi$ injective? In particular, what is the minimal number of measurements required for injectivity? When the measurement map $A_\Phi$ is stable?

(ii) For which frames $\Phi$ with injective measurement map $A_\Phi$ can $x$ be reconstructed robustly and efficiently?

In this section we give a brief overview of some of the main approaches and recent results in both research directions. We begin with discussing conditions for injectivity of the measurement map $A_\Phi$, and then describe main existing reconstruction algorithms and their modifications for different classes of frames. We also discuss the recovery guarantees and robustness to measurement noise for the described algorithms. For a more detailed overview of the recent developments in phase retrieval we refer the reader to [43] and references therein.

4.2.1 Injectivity and stability of phase retrieval

In the investigation of the injectivity of phaseless measurement maps, the question about the minimal number of measurements required receives the most attention. More precisely, the following two values are studied

$$N_{EX}(M) = \min \{ |\Phi| : A_\Phi \text{ is injective} \};$$

$$N_{ALL}(M) = \min \{ N : \text{for almost all } \Phi, \ |\Phi| \geq N, \ A_\Phi \text{ is injective} \}.$$

In words, $N_{EX}(M)$ is the minimal number, such that there exists at least one frame $\Phi \subset \mathbb{C}^M$ of cardinality $N_{EX}(M)$ with injective measurement map $A_\Phi$; and $N_{ALL}(M)$ is the minimal number, such that a generic frame $\Phi \subset \mathbb{C}^M$ of cardinality $N_{ALL}(M)$ induces an injective measurement map $A_\Phi$. The latter means that there exists an open dense subset in the space of all $N_{ALL}(M)$-element frames in $\mathbb{C}^M$ endowed with Zariski topology, such that any frame $\Phi$ in it induces an injective map $A_\Phi$.

In the real case, when we consider the restriction of $A$ to $\mathbb{R}^M$, the following criterion for injectivity is shown by Balan, Casazza, and Edidin [4].

**Theorem 4.2.1.** [4] Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{R}^M$ be a frame with corresponding measurement map $A_\Phi$. The restriction of the measurement map $A_\Phi|_{\mathbb{R}^M}$ is injective if and only if for every $T \subset \{1, \ldots, N\}$ either $\{\varphi_n\}_{n \in T}$ or $\{\varphi_n\}_{n \in \{1, \ldots, N\} \setminus T}$ spans $\mathbb{R}^M$.

The following result is a corollary of this criterion.

**Theorem 4.2.2.** [4] In the real case, for any dimension $M$,

$$N_{EX}(M) = N_{ALL}(M) = 2M - 1.$$
A similar characterization of injectivity of measurement maps in the complex setting is presented in the paper by Bandeira, Cahill, Mixon, and Nelson [5]. Unfortunately it does not allow to compute values of $N_{EX}(M)$ or $N_{ALL}(M)$, as Theorem 4.2.1 does in the real case.

**Theorem 4.2.3.** [5] Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame with corresponding measurement map $A_\Phi$. Then $A_\Phi$ is injective if and only if for every nonzero vector $u$, $\text{span} \{\varphi_n \varphi_n^* u\}_{n=1}^N = \text{span} \{iu\}^\perp$.

For the complex case, the following conjecture has been proposed by Bandeira, Cahill, Mixon, and Nelson in 2014 [5].

**Conjecture 4.2.4.** (The 4M-4 Conjecture.) For any $M \geq 2$, consider a frame $\Phi \subset \mathbb{C}^M$. Then the following holds:

(i) If $|\Phi| < 4M - 4$, then $A_\Phi$ is not injective.

(ii) If $N \geq 4M - 4$, then $A_\Phi$ is injective for a generic $\Phi$ with $|\Phi| = N$.

In short, $N_{EX} = N_{ALL} = 4M - 4$.

Over the last decade the following progress has been achieved on this conjecture.

- In 2006 Balan, Casazza, and Edidin showed that if $N \geq 4M - 2$ then $A_\Phi$ is injective for a generic $\Phi$ [4].
- In 2011 Heinosaari, Mazzarella, and Wolf managed to show that if $|\Phi| < 4M - 2\alpha(M - 1) - 3 = (4 + o(1))M$, where $\alpha(M - 1) \leq \log_2(M)$ is the number of 1’s in the binary expansion of $M - 1$, then $A_\Phi$ is not injective [39].
- In 2014 several examples of frames with cardinality $4M - 4$ and injective measurement maps were constructed [9, 29].
- Later in 2014 Conca, Edidin, Hering, and Vinzant and, independently, Király and Ehler showed that if $N \geq 4M - 4$, then $A_\Phi$ is injective for a generic $\Phi$ with $|\Phi| = N$, which proves part (ii) of the conjecture [22, 46].
- In 2015 Vinzant disproved part (i) of the conjecture for $M = 4$. She constructed a frame with 11 vectors and showed the injectivity of $A$ for this frame [77].

The example of an injective measurement frame proposed in [77] is not unique. In fact, this paper shows that the set of injective frames is of full dimension in $\mathbb{C}^{4 \times 11}$. Even though this certainly disproves part (i) of the 4M − 4 Conjecture in the case $M = 4$, Vitzant conjectured that it is asymptotically true in the following probabilistic sense.

**Conjecture 4.2.5.** (Vinzant’s Refined Injectivity Conjecture.) Draw $\Phi$ uniformly from the Grassmannian of $M$-dimensional subspaces of $\mathbb{C}^{4M-5}$. Let $p_M$ denote the probability that the measurement map $A_\Phi$ is injective. Then

(i) $p_M < 1$ for all $M$.

(ii) $\lim_{M \to \infty} p_M = 0$. 
4.2. OVERVIEW OF STATE OF THE ART IN PHASE RETRIEVAL

While part (ii) of the $4M - 4$ Conjecture, proven by Conca, Edidin et. al. (and, independently, by Király and Ehler), guarantees that for a randomly selected frame $\Phi$ with $|\Phi| \geq 4M - 4$ the measurement map is injective with probability 1, it does not provide any method to check whether the measurement map of a concrete frame is injective. Since in practice the particular structure of the frame is often dictated by the application considered, it is also important to study injectivity of $A$ for some particular classes of frames.

As such, the injectivity property of the full Gabor frames were studied by Bojarovska and Flinth [10]. In particular, they found the following easily checkable sufficient condition for injectivity.

**Theorem 4.2.6.** [10] Let $g \in \mathbb{C}^M$ be a window, such that for any $\lambda \in \mathbb{Z}_M \times \mathbb{Z}_M$

$$\langle g, \pi(\lambda)g \rangle \neq 0.$$ 

Then the measurement map $A_{(g,\mathbb{Z}_M \times \mathbb{Z}_M)}$, corresponding to the full Gabor frame $(g,\mathbb{Z}_M \times \mathbb{Z}_M)$, is injective.

**Remark 4.2.7.** Note, that the number of measurements considered in Theorem 4.2.6 is $|(g,\mathbb{Z}_M \times \mathbb{Z}_M)| = M^2$, which is too big in comparison to the dimension of the signal. Essentially, this result says that under the assumption of the theorem the system $\{(\pi(\lambda)g)(\pi(\lambda)g)^*, \lambda \in \mathbb{Z}_M \times \mathbb{Z}_M\}$ spans the space $H^M$ of Hermitian $M \times M$ matrices. Finding a condition for injectivity of phaseless measurements with respect to a Gabor frame $(g, \Lambda)$ with $|\Lambda| < M^2$ remains an important open question.

Another important research task in phase retrieval is to find conditions on the measurement frame $\Phi$ to ensure stable uniqueness of the reconstruction of a signal $x$ from its phaseless measurements $A_{\Phi}(x)$, irrespective of the specific reconstruction method used. In other words, we want to ensure that, if for two signals $x$ and $y$ the measurements $A_{\Phi}(x)$ and $A_{\Phi}(y)$ are close, then $x$ and $y$ are also close up to a global phase factor. Eldar and Mendelson proposed the following notion of phaseless measurement map stability, formalizing this idea in the real case [27].

**Definition 4.2.8.** Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{R}^M$ be a frame. The measurement map $A_{\Phi} : \mathbb{R}^M \to \mathbb{R}^N$ given by $A_{\Phi}(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$ is called stable with a constant $C$ in a set $T \subset \mathbb{R}^M$ if for every $x, y \in T$,

$$||A_{\Phi}(x) - A_{\Phi}(y)||_1 \geq C||x - y||_2||x + y||_2.$$ 

Note that stability in a set is a much stronger property than injectivity up to a global phase factor. Indeed, the latter means that $||A_{\Phi}(x) - A_{\Phi}(y)||_1 > 0$ for $x \neq \pm y$, but without any quantitative estimates on the norm of the difference. We also note that $\ell_1$ norm is a natural choice for the distance between measurements of signals, since the measurement map $A_{\Phi}$ outputs squared absolute values of frame coefficients.

It has been shown that for a frame $\Phi$ of cardinality $O(M)$, such that $\varphi_j(m)$ are independent $L$-subgaussian random variables, the mapping $A_{\Phi}$ is stable in $\mathbb{R}^M$ under the additional small ball assumption on the distribution of $\varphi_j(m)$ [27]. More precisely, Eldar and Mendelson showed the following result.
Theorem 4.2.9. [27] Let \( L > 0 \) and consider a frame \( \Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M \) with \( \varphi_j \) independent identically distributed isotropic \( L \)-subgaussian random vectors satisfying the following small ball assumption. There exists a constant \( c > 0 \) such that for every \( x \in \mathbb{R}^M \) and \( \varepsilon > 0 \)
\[
P\{|\langle \varphi_j, x \rangle| \leq \varepsilon ||x||_2\} \leq c\varepsilon.
\]
Then there exist constants \( c_1, c_2, c_3 > 0 \) depending only on \( L \) and \( c \), such that, assuming \( |\Phi| = N \geq c_1 M \), the measurement map \( A_\Phi \) is stable with the constant \( c_3 \) in \( \mathbb{R}^M \) with probability at least \( 1 - 2e^{-c_2 M} \).

Later Krahmer and Liu showed that the small ball assumption can be dropped if we consider stability in the set of all vectors \( x \) that are not too “spiky” in the sense that at most a small fraction of their norm is concentrated on a single coordinate.

Theorem 4.2.10. [47] Let \( L > 0 \) and \( 0 < \mu < \frac{1}{2\sqrt{2}} \) be arbitrary constants. Consider a frame \( \Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M \), where \( \varphi_j \) are independent identically distributed \( L \)-subgaussian random vectors with mean zero and variance \( I_M \). Then there exist constants \( c_1, c_2, c_3 > 0 \) depending only on \( L \) and \( \mu \), such that, assuming \( |\Phi| = N \geq c_1 M \), the measurement map \( A_\Phi \) is stable with the constant \( c_3 \) in the set of \( \mu \)-flat vectors \( T_\mu = \{x \in \mathbb{R}^M, ||x||_\infty \leq \mu ||x||_2\} \) with probability at least \( 1 - 2e^{-c_2 M} \).

We address the question of stability for a more general class of random frames with independent frame vectors having a bounded fourth moment and also for Gabor frames with a random window in Section 4.3.

4.2.2 Reconstruction algorithms

As discussed in Section 4.2.1, almost all frames with at least \( 4M - 4 \) vectors induce injective measurement maps. Nonetheless, this result does not provide a feasible reconstruction algorithm. Until recently, very little was known about how to achieve stable and efficient reconstruction given injectivity. Most of popular practical phase retrieval methods in use today have their origins in the alternating projection algorithms that were developed in the 1970s by Gerchberg and Saxton [35]. These algorithms are conceptually simple, efficient to implement, and, hence, popular among practitioners, despite the lack of a rigorous mathematical understanding of their properties or availability of global recovery guarantees. Indeed, given the non-convex nature of the phase retrieval problem, the lack of global convergence guarantees for these methods is not surprising. In particular, they require delicate parameter and initial guess selection to enable reconstruction, otherwise they might have multiple stationary points or a stationary point outside the non-convex set \( \{z : |F(z)(\omega)| = |F(x)(\omega)|\} \).

More recently, there have been significant efforts devoted towards developing phase retrieval algorithms which are efficient, have provable recovery guaranties, and are robust to noise in the measurements.

In this section, we describe main existing reconstruction algorithms, such as PhaseLift [17], phase retrieval with polarization [1], Wirtinger flow algorithm [15], and phase retrieval from local correlation measurements [42], and their modifications for different classes of frames. We also discuss the recovery guaranties and robustness to measurement noise established for these algorithms. Note that for most of them recovery and robustness guarantees are obtained when the measurement
frame is randomly generated with independent frame vectors. Since measurements of this type are hard to implement in practice, the design of fast and stable recovery algorithms with a small number of application relevant, structured, measurements remains an important problem.

**PhaseLift**

Let $\Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{C}^M$ be a frame, and $b$ be a vector of phaseless measurements of the unknown signal $x_0 \in \mathbb{C}^M$ with $\Phi$. Then the phase retrieval problem can be formulated as follows

$$\begin{align*}
\text{find} & \quad x \\
\text{subject to} & \quad A_\Phi(x) = b.
\end{align*}$$

The PhaseLift algorithm is based on a lifting trick, which was proposed by Balan, Bodmann, Casazza, and Edidin [3]. The key idea is that the intensity measurements can be lifted into the $M^2$-dimensional real vector space $H^M$ of self-adjoint operators, where measurements have the form of Hilbert-Schmidt inner products. We have

$$A_\Phi(x)(j) = |\langle \varphi_j, x \rangle|^2 = \text{Tr}(x^* \varphi_j \varphi_j^* x) = \text{Tr}(\varphi_j \varphi_j^* xx^*) = \text{Tr}(A_j X) = \langle X, A_j \rangle_{HS},$$

where $X = xx^*$ and $A_j = \varphi_j \varphi_j^*$ are rank-one positive semidefinite matrices. Then the initial phase retrieval problem is equivalent to the following rank minimization problem

$$\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad \langle X, A_j \rangle_{HS} = b_j, \ j \in \{1, \ldots, N\}, \\
& \quad X \succeq 0.
\end{align*}$$

It is clear that if $N = M^2$ and provided the lifted measurement vectors $A_j$ are linearly independent in $H^M$, problem (4.2) becomes a feasibility problem, and $X$ can be recovered exactly and efficiently. But, in this case, the number of measurements $N = M^2$ is too high in comparison to the ambient dimension. Thus we are interested in the case when $N < M^2$, and (4.2) becomes an NP-hard problem, since $\text{rank}(\cdot)$ is a non-convex functional.

Inspired by this lifting trick, Candès, Eldar, Strohmer, and Voroninski used $\text{Tr}(\cdot)$ as a convex surrogate for $\text{rank}(\cdot)$ and proposed the following convex relaxation of (4.2) [12, 17].
minimize \( \text{Tr}(X) \)
subject to \( \text{Tr}(A_jX) = b_j, \ j \in \{1, \ldots, N\} \), \( X \succeq 0 \). (4.3)

This semidefinite program is known as PhaseLift.

It has been shown that if the measurement vectors \( \varphi_j \) are independent Gaussian vectors and \( N = O(M) \), then the solution of the convex relaxation (4.3) is exact [13]. More precisely, the following result holds.

**Theorem 4.2.11.** [13] Let \( N \geq c_0 M \), where \( c_0 \) is a sufficiently large constant, and consider frame \( \Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{C}^M \), such that \( \varphi_j \sim \text{i.i.d. } \mathcal{CN}(0, I_M) \). Then there exists a numerical constant \( \gamma > 0 \), such that, for all \( x_0 \in \mathbb{C}^M \), with probability at least \( 1 - O(e^{-\gamma N}) \) (4.3) has the unique solution \( x_0x_0^* \).

In many applications measurements are corrupted by noise and have the form
\[
\tilde{b}_j = |\langle \varphi_j, x \rangle|^2 + \eta_j, \ j \in \{1, \ldots, N\},
\]
where \( \eta_j \) are the noise terms. In this case, one can find an approximation of the signal using the following modification of the PhaseLift algorithm [13].

minimize \( \sum_{1 \leq j \leq N} |\text{Tr}(A_jX) - b_j| \)
subject to \( X \succeq 0 \). (4.4)

**Theorem 4.2.12.** [13] Consider the setup of Theorem 4.2.11. Then there exist numerical constants \( \gamma > 0 \) and \( C_0 \), such that for all \( x_0 \in \mathbb{C}^M \), the solution \( \hat{X} \) of (4.4) with probability at least \( 1 - O(e^{-\gamma N}) \) obeys
\[
||\hat{X} - x_0x_0^*||_F \leq C_0 \frac{||\eta||_1}{N}.
\]

Here \( ||\cdot||_F \) denotes the Frobenius norm and \( ||\cdot||_1 \) denotes the \( \ell^1 \)-norm. By finding the eigenvector \( \hat{x} \) of \( \hat{X} \) with the largest eigenvalue, one has, with the same probability,
\[
\min_{\phi \in [0, 2\pi)} ||\hat{x} - e^{i\phi}x_0||_2 \leq C_0 \min \left( ||x_0||_2, \frac{||\eta||_1}{N||x_0||_2} \right).
\]

While the above results only consider randomly generated frames \( \{\varphi_j\}_{j=1}^N \) with independent \( \varphi_j \)'s, derandomization of PhaseLift is an important task which currently receives a lot of attention. A partial derandomization has been proposed by Candes, Li, and Soltanolkotabi [14]. They consider measurements of the form of coded diffraction patterns \( \{|f_k^\ell D_\ell x\}|_{k \in \{1, \ldots, M\}, \ell \in \{1, \ldots, L\}} \), where \( f_k^\ell \) are the rows of the Fourier matrix and \( D_\ell \) are random diagonal matrices modelling the masks. They showed that a signal \( x_0 \) can be recovered exactly (up to a global phase factor) from the suggested measurements using PhaseLift as long as the number of masks is \( L = O(\log^3 M) \), implying the total number of measurements to be \( O(M \log^4 M) \).
Theorem 4.2.13. [14] Let $\xi$ be a complex symmetric random variable, satisfying the following moment assumptions

$$\mathbb{E}\xi = 0; \quad \mathbb{E}\xi^2 = 0; \quad \mathbb{E}|\xi|^4 = 2\mathbb{E}|\xi|^2.$$  

Furthermore, let $D_\ell$, $\ell \in \{1, \ldots, L\}$, be independent copies of a diagonal matrix $D$, whose diagonal entries are i.i.d. copies of the random variable $\xi$. If $L \geq c \log^4(M)$ for some numerical constant $c$, then, for every $x_0 \in \mathbb{C}^M$, with probability at least $1 - 1/M$ problem (4.3) has the unique solution $x_0 x_0^*$.  

The number of coded diffraction patterns with random masks required for reconstruction has been further reduced by Gross, Krahmer, and Kueng. They showed that $O(M \log^2 M)$ measurements of this type are sufficient for the reconstruction while a lower bound for the number of masks required to allow for recovery with any algorithm is given by $O(\log M)$ [37]. Unfortunately, this modification of PhaseLift has not been shown to be robust to noise. Numerical experiments illustrating this approach, though, demonstrate its effectiveness and stability [14]. Another partial derandomization of PhaseLift using spherical designs has been obtained in [36].  

Recently, a complete derandomization of PhaseLift has been suggested by Kech [45]. More precisely, he proved the following result.

Theorem 4.2.14. [45] Let us fix $\{x_k\}_{k=1}^{2M-3} \subset \mathbb{R}$, such that $x_1 < x_2 < \cdots < x_{2M-3}$, and define vectors $v_k \in \mathbb{C}^M$ by

$$v_k = (1, x_k e^{i\pi/2}, x_k^2 e^{i\pi/2}, \ldots, x_k^{M-1} e^{(M-1)i\pi/2})^T.$$  

Furthermore, denote by $\{e_j\}_{j=1}^M$ the standard basis in $\mathbb{C}^M$, and define a frame

$$\Phi = \left\{ e_1, e_2, \ldots, e_M, \frac{v_1}{||v_1||}, \frac{v_2}{||v_1||}, \ldots, \frac{v_{2M-3}}{||v_{2M-3}||}, \frac{\bar{v}_{2M-3}}{||v_{2M-3}||} \right\}.$$  

Then, for every $x \in \mathbb{C}^M$, the unique minimizer of (4.3) with $b = A_\Phi(x)$ is $\hat{X} = xx^*$.  

Note that, unlike Theorem 4.2.11 and Theorem 4.2.13, which guaranty the recovery only with a certain probability, this result is completely deterministic and thus guaranties recovery at all times. Furthermore, it uses $5M - 6$ measurements which is very close to the theoretical bound $(4 + o(1))M$.

Phase retrieval with polarization

Here, we describe the polarization approach to phase retrieval that has been proposed by Alexeev, Bandeira, Fickus, and Mixon [1, 6]. One of the main differences between PhaseLift and polarization is that polarization requires measurement vectors to have a particular structure. If a phase retrieval application allows one to design measurement vectors with the required structure, then the initial signal can be recovered using polarization approach very efficiently. A number of numerical simulations in [1, 6] that compare polarization to PhaseLift show that polarization is much faster, that is, seconds versus hours, while PhaseLift often offers more stable estimates of a signal.
The polarization approach to phase retrieval can be described as follows, see also [1]. Suppose $\Phi_V = \{\varphi_j\}_{j \in V} \subset \mathbb{C}^M$ is a measurement frame. We consider the phase retrieval problem

$$\text{find } x \text{ subject to } |\langle x, \varphi_j \rangle|^2 = b_j.$$  \hfill (4.5)

For any $(i, j) \in V \times V$ with $|\langle x, \varphi_i \rangle| \neq 0$ and $|\langle x, \varphi_j \rangle| \neq 0$, we define the relative phase between frame coefficients as

$$\omega_{ij} = \left( \frac{|\langle x, \varphi_i \rangle|}{|\langle x, \varphi_j \rangle|} \right)^{-1} \frac{\langle x, \varphi_j \rangle}{\langle x, \varphi_i \rangle} = \frac{\langle x, \varphi_i \rangle \langle x, \varphi_j \rangle}{|\langle x, \varphi_i \rangle||\langle x, \varphi_j \rangle|}. \hfill (4.6)$$

Note that $\omega_{ij} \omega_{jk} = \omega_{ik}$. Suppose that we are given $\{\omega_{ij}\}_{(i,j) \in E}$ for some set $E \subset V \times V$ in addition to the phaseless measurements with respect to $\Phi_V$. Then we seek to solve the following simpler problem

$$\text{find } x \text{ subject to } \frac{\langle x, \varphi_i \rangle \langle x, \varphi_j \rangle}{|\langle x, \varphi_i \rangle||\langle x, \varphi_j \rangle|} = \omega_{ij}, \ (i, j) \in E; \hfill (4.7)$$

$$|\langle x, \varphi_i \rangle|^2 = b_i, \ i \in V.$$  

This problem can be solved using phase propagation. More precisely, we choose $i_0 \in V$, such that $|\langle x, \varphi_{i_0} \rangle| \neq 0$, set $c_{i_0} = |\langle x, \varphi_{i_0} \rangle|$, and for every $j \in V$ with $(i_0, j) \in E$ define

$$c_j = \begin{cases} 
\omega_{i_0 j} |\langle x, \varphi_j \rangle| & \text{if } |\langle x, \varphi_j \rangle| \neq 0, \\
0 & \text{otherwise.}
\end{cases}$$

In the next step, for each $k$ with $c_k$ not defined yet and $(i_0, j), (j, k) \in E$ for some $j$ with $b_j \neq 0$, we set

$$c_k = \begin{cases} 
\omega_{j k} c_j |\langle x, \varphi_k \rangle| & \text{if } |\langle x, \varphi_k \rangle| \neq 0, \\
0 & \text{otherwise.}
\end{cases}$$

We repeat this step iteratively until values $c_i$ are assigned to all indices $i \in V$ that can be reached from $i_0$ using edges from $E$. This process is described in Algorithm 2 and illustrated in Figure 4.2 (left).
Algorithm 2: Phase propagation algorithm

\textbf{Input :} measurements \( \{ b_i = |\langle x, \varphi_i \rangle|^2 \} \), \( \varphi_i \in \Phi_V \), \( \{ \omega_{ij} = \frac{\langle x, \varphi_i \rangle \langle x, \varphi_j \rangle}{|\langle x, \varphi_i \rangle||\langle x, \varphi_j \rangle|} \} \) \( (i,j) \in E \).

\textbf{Output:} \( x = \sum_{j \in V} c_j \tilde{\varphi}_j \).

1. choose \( i_0 \), such that \( b_{i_0} \neq 0 \), set \( c_{i_0} = \sqrt{b_{i_0}} \);
2. for all \( j \), s.t. \( b_j = 0 \), set \( c_j = 0 \);
3. while not all \( c_j \) set do
   4. choose \( c_i \neq 0 \) already known;
   5. for all \( j \), s.t. \( (i,j) \in E \) and \( c_j \) is not set do
      6. \( c_j = \omega_{ij} \frac{c_i}{|c_i|} \sqrt{b_j} \).
   7. end
8. end

Assume that we were able to compute \( c_i \) for all \( i \in V \). Then, using a dual frame \( \tilde{\Phi}_V = \{ \tilde{\varphi}_i \}_{i \in V} \) and treating \( c_i \)’s as frame coefficients, we reconstruct a representative of the “up-to-a-global-phase” equivalence class \( [x] \) as

\[ \sum_{j \in V} c_j \tilde{\varphi}_j = \sum_{j \in V} \omega_{ij} \langle x, \varphi_j \rangle \langle \tilde{\varphi}_j \rangle = (\frac{\langle x, \varphi_i \rangle}{|\langle x, \varphi_i \rangle|})^{-1} \sum_{j \in V} \langle x, \varphi_j \rangle \langle \tilde{\varphi}_j \rangle = (\frac{\langle x, \varphi_i \rangle}{|\langle x, \varphi_i \rangle|})^{-1} x \in [x]. \]

Let us consider the graph \( G = (V, E) \), later called the graph of measurements, with the set of vertices indexed by \( V \) and the set of edges \( E \). From the phase propagation procedure, it is apparent that if \( \langle x, \varphi_j \rangle = 0 \) for some \( j \in V \), then the corresponding relative phases \( \omega_{ji} \) are not defined for all \( i \in V \) and the phase cannot be propagated through vertex \( j \). This has the effect of deleting vertex \( j \) from \( G \), see Figure 4.2 (right). If \( G \) remains connected after deleting all “zero” vertices, then, for every vertex \( i \), there exists a path from \( i_0 \) to \( i \), and \( c_i \) can be computed. This solves problem (4.7).

Thus, the initial phase retrieval problem (4.5) is reduced to the problem of finding relative phases between pairs of frame coefficients from a set \( E \), such that the corresponding graph of measurements \( G = (V, E) \) satisfies strong connectivity properties (as expander graphs do, see Section 2.3). To obtain the relative phase between frame coefficients, the following form of polarization identity is useful.

**Lemma 4.2.15.** [1] Let \( \omega = e^{2\pi i / 3} \). If \( \langle x, \varphi_i \rangle \neq 0 \) and \( \langle x, \varphi_j \rangle \neq 0 \), then

\[ \omega_{ij} = \frac{1}{3|\langle x, \varphi_i \rangle||\langle x, \varphi_j \rangle|} \sum_{k=0}^{2} \omega^k |\langle x, \varphi_i + \omega^k \varphi_j \rangle|^2. \]

In other words, to compute the relative phase \( \omega_{ij} \) between the nonzero frame coefficients \( \langle x, \varphi_i \rangle \) and \( \langle x, \varphi_j \rangle \), we use three additional phaseless measurements of \( x \) with respect to \( \varphi_i + \varphi_j, \varphi_i + \omega \varphi_j, \) and \( \varphi_i + \omega^2 \varphi_j \). This means that reconstruction of \( x \) using phase propagation involves only phaseless measurements, namely, phaseless measurements with respect to the union \( \Phi_V \cup \Phi_E \), where \( \Phi_V \) is a “vertex” frame and \( \Phi_E = \{ \varphi_i + \omega^k \varphi_j \}_{(i,j) \in E, k \in \{0,1,2\}} \). Note that \( |\Phi_V \cup \Phi_E| = |V| + 3|E| \).

Alexeev, Bandeira, Fickus, and Mixon showed that in the noiseless case one can perform the phase retrieval with polarization using only \( O(M) \) measurements [1]. They also showed that the constructed algorithm with some modifications is robust.
to noise provided $\Phi_V$ consists of independent Gaussian vectors and the number of measurements is $O(M \log M)$.

**Theorem 4.2.16.** [1] Let $N \geq CM \log M$ with $C$ sufficiently large, and consider a frame $\Phi_V \cup \Phi_E$, where $\Phi_V = \{\varphi_j\}_{j=1}^N$ with $\varphi_j$ independent standard Gaussian vectors, and $\Phi_E$ constructed as above. Then there exist constants $C', K > 0$ such that the following guarantee holds for all $x \in \mathbb{C}^M$ with overwhelming probability.

Consider noisy measurements of the form $b_j = |\langle x, \varphi_j \rangle|^2 + \eta_j$, where noise vector $\eta$ satisfies $||\eta||_2 \leq \frac{C'}{\sqrt{M}}$. Then phase retrieval with polarization produces an estimate $\tilde{x}$, such that

$$\min_{\theta \in [0, 2\pi)} ||x - e^{i\theta} \tilde{x}||_2 \leq K \sqrt{\frac{M}{\log M}} ||\eta||_2.$$  

In [6] Bandeira, Chen, and Mixon adapted the polarization method to work with measurements of the form of coded diffraction patterns. Using additive combinatorics tools, the authors showed that the graph of measurements they are using for reconstruction is sufficiently connected provided that the total number of measurements is $O(M \log M)$. However, no robustness results were obtained for polarization in the case of structured measurements before.

In Section 4.4 we use the idea of polarization to build a recovery algorithm for time-frequency structured measurements and show reconstruction and robustness guarantees for the designed algorithm in Section 4.5.

**Wirtinger flow**

A different approach to phase retrieval based on non-convex optimization has been proposed by Candes, Li, and Soltanolkotabi [15].

Let $\ell(x, y) = (x - y)^2$ be a quadratic loss function measuring the misfit between two scalar arguments. Then the solution to the phase retrieval problem with a measurement frame $\Phi = \{\varphi_i\}_{i=1}^N \subset \mathbb{C}^M$ can be obtained by solving

$$\min_{z \in \mathbb{C}^M} f(z) = \frac{1}{2N} \sum_{i=1}^N \ell(b_i, |\langle z, \varphi_i \rangle|^2),$$  

where $f : \mathbb{C}^M \to \mathbb{R}^N$ is not holomorphic, (4.9) can still be viewed as a gradient descent based on Wirtinger derivatives.

The algorithm, proposed in [15], is called *Wirtinger flow algorithm* and has two components.
4.2. OVERVIEW OF STATE OF THE ART IN PHASE RETRIEVAL

(i) A careful initialization obtained by means of spectral method. The initial guess $z_0$ is computed as the eigenvector corresponding to the largest eigenvalue of

$$ Y = \frac{1}{N} \sum_{i=1}^{N} b_i \varphi_i \varphi_i^*, $$

normalized so that $||z_0||_2 = \sqrt{\frac{M}{\sum_{i=1}^{N} ||\varphi_i||_2^2}}$.

(ii) A series of updates refining the initial estimate by the iterative update rule (4.9).

The following recovery result is proven in [15].

**Theorem 4.2.17.** [15] Let $x \in \mathbb{C}^M$ and $b = A \Phi(x)$, where $\Phi$ is a random Gaussian frame with $|\Phi| = N \geq C_0 M \log M$ for a sufficiently large numerical constant $C_0 > 0$. Assume that $\mu \leq \frac{C_1}{M}$ for some $C_1 > 0$. Then there exist a numerical constant $\gamma > 0$ and an event of probability at least $1 - 13e^{-\gamma M} - N e^{-1.5N} - 8/M^2$, such that on this event, starting with the initial guess $z_0$ given by (i), we have

$$ \min_{\theta \in [0, 2\pi]} ||x - e^{i\theta} z_\tau||_2 \leq \frac{1}{8} \left( 1 - \frac{\mu}{4} \right)^{\tau/2} ||x||_2, $$

for every $\tau > 0$.

With a similar idea in mind, several modifications of the Wirtinger flow algorithm have been constructed. Among such modifications are truncated Wirtinger flow [19], reshaped Wirtinger flow [83], and reweighted Wirtinger flow methods [82].

**Local correlation measurements**

A fast phase retrieval algorithm from local correlation measurements have been proposed in [42]. It is based on the following idea. Suppose that we are solving the phase retrieval problem with the measurement map given by

$$ A(x) = |Ax|^2 + \eta, $$

where $A \in \mathbb{C}^{N \times M}$ is the measurement matrix, $\eta$ is a noise term, and $|\cdot|^2$ denotes the pointwise squared absolute value. Furthermore, assume that $A$ has $N = (2\delta - 1)M$ rows corresponding to local correlation-based measurements, where $\delta \in \mathbb{N}$ represents the support size of the associated correlation masks. In other words, $A$ can be decomposed into $2\delta - 1$ blocks $A_1, A_2, \ldots, A_{2\delta-1} \in \mathbb{C}^{M \times M}$

$$ A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2\delta-1} \end{pmatrix}, $$

so that $A_\ell$ are circulant matrices given by $(A_\ell)_{ij} = (a_\ell)_{(j-i) \mod M+1}$, and masks $a_\ell$ are such that $(a_\ell)_i = 0$ for all $i > \delta$ and $1 \leq \ell \leq 2\delta - 1$.

As a consequence of this structure, the squared magnitude measurements from the $\ell$th block can be written as

$$ (|A_\ell x|^2)_i = (A_\ell x)_i (\overline{A_\ell x})_i = \sum_{j,k=1}^{\delta} (a_\ell)_j (a_\ell)_k \bar{x}_{i+j-1} x_{i+k-1}. $$
Let us define $y \in \mathbb{C}^N$ pointwise by

$$y_i = \bar{x}[i+\delta-1]x[i+\delta-1] + ((i+\delta-2) \mod (2\delta-1)) - \delta + 1, \quad i \in \{1, \ldots, N\}.$$ 

Then there exists a permutation matrix $P \in \{0, 1\}^{N \times N}$, such that

$$P|Ax|^2 = A' y = \begin{pmatrix} A'_1 & A'_2 & \cdots & A'_\delta & 0 & 0 & \cdots & 0 \\ 0 & A'_1 & A'_2 & \cdots & A'_\delta & 0 & \cdots & 0 \\ & & & & & & & \\ \cdots & & & & & & & \\ A'_2 & \cdots & A'_\delta & 0 & \cdots & 0 & \cdots & A'_1 \end{pmatrix} y,$$

where blocks $A'_1, A'_2, \ldots, A'_\delta \in \mathbb{C}^{(2\delta-1) \times (2\delta-1)}$ have entries

$$(A'_i)_{ij} = \begin{cases} \overline{(a_i)_{\ell}} (a_i)_{\ell+j-1}, & \text{if } 1 \leq j \leq \delta - \ell + 1; \\ 0, & \text{if } \delta - \ell + 2 \leq j \leq 2\delta - \ell - 1; \\ (a_i)_{\ell+1} (a_i)_{\ell+j-2\delta+1}, & \text{if } 2\delta - \ell \leq j \leq 2\delta - 1 \text{ and } \ell < \delta; \\ 0, & \text{if } j > 1 \text{ and } \ell = \delta. \end{cases}$$

Let $b = A(x)$. The reconstruction algorithm proposed in [42] consists of two steps. In the first step we find an approximation of vector $y$ as

$$\tilde{y} = (A')^{-1} Pb = y + (A')^{-1} P \eta.$$ 

And in the second step we use angular synchronization (see Section 4.5, Algorithm 5 for the description) to reconstruct an approximation $\tilde{x}$ of $x$ from $\tilde{y}$. The following recovery guarantees are proven in [42].

**Theorem 4.2.18.** [42] For $M$ sufficiently large, let $x \in \mathbb{C}^M$ be such that

$$||x||_2^2 \geq C(M \log M)^2 \log^3(\log M)||\eta||_2.$$ 

Then one can select a random matrix $A \in \mathbb{C}^{N \times M}$ of the form as described above and with $N = O(M \log^2 M \log^3(\log M))$, such that with probability at least $1 - \frac{1}{C' \log^* M \log^3(\log M)}$ the algorithm described above outputs $\hat{x}$ with

$$\min_{\theta \in [0,2\pi]} ||x - e^{i\theta} \hat{x}||_2^2 \leq C''(M \log M)^2 \log^3(\log M)||\eta||_2^2,$$

for some numerical constants $C', C''$. Furthermore, the algorithm runs in $O(M \log^3 M \log^3(\log M))$ time.

Iwen, Preskitt, Saab, and Viswananathan improved this approach and proposed a modified algorithm that admits deterministic measurement constructions and provides better robustness and recovery guarantees [41]. Their main result is formulated as follows.

**Theorem 4.2.19.** [41] For $M$ sufficiently large, let $x \in \mathbb{C}^M$ be an unknown signal. Let $x_{\text{min}} = \min_{j \in \{1, \ldots, M\}} |(x_0)_j|$ be the smallest magnitude of all entries of $x$. Then the modified phase retrieval algorithm outputs $\hat{x}$ with

$$\min_{\theta \in [0,2\pi]} ||x - e^{i\theta} \hat{x}||_2 \leq C \left( \frac{||x||_\infty}{x_{\text{min}}} \right) \left( \frac{M}{\delta} \right) \kappa ||\eta||_2 + CM^{\frac{1}{2}} \sqrt{\kappa ||\eta||_2}.$$ 

Here, $C > 0$ is some universal constant, $\delta$ is the number of masks used, and $\kappa$ depends on the condition number of the matrix constructed similarly to (4.10).
4.3 Stability of phase retrieval using frame order statistics

As discussed in Section 4.2.1, an important problem in phase retrieval is to establish for which measurement frames \( \Phi = \{ \varphi_j \}_{j=1}^N \subset \mathbb{R}^M \) the signal \( x \in \mathbb{R}^M \) can be stably reconstructed from its phaseless measurements \( A_\Phi(x) = \{|\langle x, \varphi_j \rangle|\}_{j=1}^N \). In other words, we want to ensure that, if for two signals \( x \) and \( y \) the measurements \( A_\Phi(x) \) and \( A_\Phi(y) \) are close, then \( x \) and \( y \) are also close up to a global phase factor. Based on this idea, Eldar and Mendelson proposed a notion of phaseless measurement map stability, irrespective of the specific reconstruction method used, see Definition 4.2.8 and [27]. More precisely, we call the measurement map \( A_\Phi \) stable with a constant \( C \) in a set \( T \subset \mathbb{R}^M \) if for every \( x, y \in T \),

\[
||A_\Phi(x) - A_\Phi(y)||_1 \geq C||x - y||_2 ||x + y||_2.
\]

Eldar and Mendelson also showed that for a frame \( \Phi \) of cardinality \( O(M) \), such that \( \varphi_j(m) \) are independent \( L \)-subgaussian random variables, the mapping \( A_\Phi \) is stable in \( \mathbb{R}^M \) under the additional small ball probability assumption on the distribution of \( \varphi_j(m) \), see Theorem 4.2.9.

The study of frame order statistics, defined in Section 3.2, is not only of interest in frame theory, but it also plays an important role in various areas of signal processing, such as phase retrieval [1, 65] and quantization [54, 20]. In this section we discuss the role of frame order statistics in the investigation of stability of the phase retrieval problem for two classes of measurement frames, namely, random frames with independent frame vectors satisfying certain moment conditions, and Gabor frames with random windows.

4.3.1 Stability for random frames with bounded fourth moment

Here, we use the bounds of the uniform frame order statistics obtained in Theorem 3.2.7 to show stability of the phaseless measurement map for a random frame with independent frame vectors under bounded fourth moment assumption. At the cost of slight increase of the measurement frame cardinality, Theorem 3.2.7 allows to show stability of \( A_\Phi \) in \( \mathbb{R}^M \) for a larger class of random frames \( \Phi \) than considered in Theorem 4.2.9, and without any additional restrictions on the set \( T \) of the measured signals, like in Theorem 4.2.10. More precisely, we have the following result.

**Theorem 4.3.1.** Let the frame \( \Phi = \{ \varphi_j \}_{j \in \{1, \ldots, N\}} \subset \mathbb{R}^M \), with \( M \) large enough, be such that \( \varphi_j(m) \), \( j \in \{1, \ldots, N\}, m \in \mathbb{Z}_M \), are independent identically distributed centered random variables, normalized so that \( \text{Var}(\varphi_j(m)) = \frac{1}{M} \). Assume further that \( \mathbb{E}(|\varphi_j(m)|^4) \leq \frac{B}{M^2} \), for some constant \( B \geq 1 \), and \( N \geq C_0 M \log M \), for some constant \( C_0 > 1 \). Consider the phaseless measurement map \( A_\Phi : \mathbb{R}^M \to \mathbb{R}^N \) given by \( A_\Phi(x) = \{|\langle x, \varphi_j \rangle|\}_{j=1}^N \). Then there exists a numerical constant \( L > 0 \), such that, with overwhelming probability, \( A_\Phi : \mathbb{R}^M \to \mathbb{R}^N \) is stable with constant \( C \geq L \log(M) \) in \( \mathbb{R}^M \) in the sense of Definition 4.2.8. In other words, for any \( x, y \in \mathbb{R}^M \),

\[
||A_\Phi(x) - A_\Phi(y)||_1 \geq L \log(M)||x - y||_2 ||x + y||_2.
\]
Proof. Let $\Phi$ and the corresponding measurement map $A_\Phi(x)$ be defined as above. Then, for any $x, y \in \mathbb{R}^M$, we have

$$||A_\Phi(x) - A_\Phi(y)||_1 = \sum_{i=1}^N ||\langle x, \varphi_i \rangle^2 - ||\langle y, \varphi_i \rangle^2|| = \sum_{i=1}^N |\langle x - y, \varphi_i \rangle||\langle x + y, \varphi_i \rangle|$$

$$= ||x - y||_2 ||x + y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - y}{||x - y||_2}, \varphi_i \right\rangle, \left\langle \frac{x + y}{||x + y||_2}, \varphi_i \right\rangle \right|.$$ 

Let us fix some $\frac{1}{2} < \alpha < 1 - \frac{1}{2^M}$. Then the real case version of Theorem 3.2.7(a) (see Remark 3.2.8) implies that there exist constants $c, c_1 > 0$, such that the following holds with probability at least $1 - e^{-c_1 M \log M}$. For every unit norm vector $u \in \mathbb{S}^{M-1}$, there exists a set of indices $J_u \subset \{1, \ldots, N\}$ with $|J_u| \geq \alpha N$, such that $|\langle u, \varphi_j \rangle| \geq \frac{c\alpha N}{\sqrt{M}}$ for all $j \in J_u$. In particular, for $u = \frac{x - y}{||x - y||_2}$ and $v = \frac{x + y}{||x + y||_2}$, there exist $J_u, J_v \subset \{1, \ldots, N\}$, such that $|J_u| \geq \alpha N$, $|J_v| \geq \alpha N$, and $|\langle u, \varphi_i \rangle||\langle v, \varphi_i \rangle| \geq \frac{c^2}{M}$ for all $j \in J_u \cap J_v$. Then, since $|J_u \cap J_v| \geq (2\alpha - 1)N > 0$, we have

$$||A_\Phi(x) - A_\Phi(y)||_1 = ||x - y||_2 ||x + y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - y}{||x - y||_2}, \varphi_i \right\rangle, \left\langle \frac{x + y}{||x + y||_2}, \varphi_i \right\rangle \right|$$

$$\geq \frac{c^2 (2\alpha - 1)N}{M} ||x - y||_2 ||x + y||_2 \geq c^2 C_0 (2\alpha - 1) \log M ||x - y||_2 ||x + y||_2,$$

for all pairs $x, y \in \mathbb{R}^M$, with probability at least $1 - e^{-c_1 M \log M}$. 

4.3.2 Stability of Gabor frames with a random window

Theorem 3.2.2 implies the following result, showing non-uniform stability of the measurement map for Gabor frames with a random window.

**Theorem 4.3.2.** Let $(g, \Lambda) \subset \mathbb{R}^M$ be a Gabor frame with a random window $g$ uniformly distributed on the real unit sphere $\mathbb{S}^{M-1}$. Define the measurement map $A_\Lambda : \mathbb{R}^M \to \mathbb{R}^M$ by $A_\Lambda(x) = \{||\langle x, \pi(\lambda)g \rangle||\}_{\lambda \in \Lambda}$. Then, for any $k > \sqrt{2}$, there exists a constant $C = \frac{1}{\pi k^2} (1 - \frac{2}{\pi k^2})$, such that for each pair $x, y \in \mathbb{R}^M$ the following holds with probability at least $1 - \frac{2}{k^2}$

$$||A_\Lambda(x) - A_\Lambda(y)||_1 \geq C ||x - y||_2 ||x + y||_2.$$

**Proof.** Consider a pair $x, y \in \mathbb{R}^M$. We have

$$||A_\Lambda(x) - A_\Lambda(y)||_1 = \sum_{\lambda \in \Lambda} ||\langle x, \pi(\lambda)g \rangle||^2 - ||\langle y, \pi(\lambda)g \rangle||^2$$

$$= \sum_{\lambda \in \Lambda} ||\langle x - y, \pi(\lambda)g \rangle|| ||\langle x + y, \pi(\lambda)g \rangle||$$

$$= ||x - y||_2 ||x + y||_2 \sum_{\lambda \in \Lambda} ||\langle u, \pi(\lambda)g \rangle|| ||\langle v, \pi(\lambda)g \rangle||,$$

where $u = \frac{x - y}{||x - y||_2}$ and $v = \frac{x + y}{||x + y||_2}$ are unit norm vectors. For any $k > 0$, let us fix some $c > 0$ sufficiently small, so that $c^2 + kc < \frac{1}{2}$. Then the real case version of Theorem 3.2.2(a) (see Remark 3.2.3) implies the following. For the unit vector $u,$
with probability at least \(1 - \frac{1}{k^2}\), there exists a set of indices \(J_u \subset \Lambda\) with cardinality \(|J_u| \geq (1 - c^2 - kc)|\Lambda|\), such that \(|\langle u, \pi(\lambda) g \rangle| \geq \frac{c}{\sqrt{M}}\) for every \(\lambda \in J_u\). The same is true for the unit vector \(v\). Thus, with probability at least \(1 - \frac{2}{k^2}\) there exists a set of indices \(J \supset J_u \cap J_v\) with \(|J| \geq (1 - 2(c^2 + kc))|\Lambda|\), such that \(|\langle u, \pi(\lambda) g \rangle||\langle v, \pi(\lambda) g \rangle| \geq \frac{c^2}{M}\) for every \(\lambda \in J\). That is, with probability at least \(1 - \frac{2}{k^2}\),

\[
||A_\Lambda(x) - A_\Lambda(y)||_1 = ||x - y||_2||x + y||_2 \sum_{\lambda \in \Lambda} |\langle u, \pi(\lambda) g \rangle||\langle v, \pi(\lambda) g \rangle| \\
\geq \frac{c^2|J|}{M}||x - y||_2||x + y||_2 \geq C||x - y||_2||x + y||_2,
\]

where \(C = c^2(1 - 2(c^2 + kc))\). Then, taking \(c = \frac{1}{3k}\) we conclude the proof.

To the best of our knowledge, only injectivity (up to a global phase) of the measurement map \(A_{\mathbb{Z}_M \times \mathbb{Z}_M}\) corresponding to a full Gabor frame \((g, \mathbb{Z}_M \times \mathbb{Z}_M)\), depending on the window \(g\), has been studied before [10]. We note that stability in a set is a much stronger property than injectivity up to a global phase factor. Indeed, the latter means that \(||A_\Phi(x) - A_\Phi(y)||_1 > 0\) for \(x \neq \pm y\), but without any quantitative estimates on the norm of the difference.

Should Conjecture 3.2.9, suggesting bounds on the uniform frame order statistics of Gabor frames, be true, stability, and thus also injectivity, of \(A_\Lambda\) with a random Gabor window \(g\) would follow for any \(\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M\) with \(|\Lambda| = O(M(\log M)^\gamma)\). This would be a big step forward in the study of phase retrieval with Gabor frames.

### 4.4 Recovery algorithm for time-frequency measurements

As discussed before, the structure of the measurement frame \(\Phi\) is usually dictated by a concrete application where the phase retrieval problem arises. For instance, measurements arising in optics and diffraction imaging are of the form of poinwise squared absolute values of masked Fourier transforms of the object, or coded diffraction patterns. In some other applications, including speech recognition and radars, the arising measurement frames are Gabor frames (see Section 2.1 for the definition and background on Gabor frames).

In this section, we design a phase retrieval algorithm for the case when the measurement frame has time-frequency structure. The main motivation for considering such frames is that, in this case, the frame coefficients are of the form of masked Fourier coefficients, where the masks are (time) shifts of the Gabor window, which can be random. This makes measurements implementable in many applications, including diffraction imaging and speech recognition, but at the same time we preserve the flexibility of the frame-theoretic approach.

We use the idea of polarization which we described in details in Section 4.2.2, see also [1, 6]. We start by describing the structure of measurement frames considered and then propose a reconstruction algorithm for the case when measurements are exact and not corrupted by noise. We postpone the discussion of the noisy case and robustness analysis of the proposed algorithm till Section 4.5.
4.4.1 Measurement process and frame construction

For an unknown signal \( x \in \mathbb{C}^M \), consider the phase retrieval problem

\[
\text{find } x \quad \text{subject to} \quad |\langle x, \varphi_j \rangle|^2 = b_j, \quad \varphi_j \in \Phi,
\]

where \( \Phi \) is a measurement frame such that

\[ \Phi = \{ \varphi_j \}_{j=1}^N = \Phi_\Lambda \cup \Phi_E \subset \mathbb{C}^M. \]

Here \( \Phi_\Lambda = (g, \Lambda) \), \( \Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M \), is a principle measurement frame which we choose to be a Gabor frame, and \( \Phi_E = \{ \pi(\lambda_1)g + e^{2\pi it/3} \pi(\lambda_2)g \}_{(\lambda_1, \lambda_2) \in E, \ t \in \{0,1,2\}} \), \( E \subset \Lambda \times \Lambda \), is a set of vectors to be used for additional measurements.

Following the idea of the polarization approach described in Section 4.2.2, we aim to use vectors from the set \( \Phi_E \) to compute relative phases (4.6) between frame coefficients of \( x \) with respect to \( \Phi_\Lambda \). Namely, using the polarization identity in Lemma 4.2.15 with \( \omega = e^{2\pi i/3} \), we obtain

\[
\omega_{\lambda_1, \lambda_2} = (\frac{\langle x, \pi(\lambda_1)g \rangle}{|\langle x, \pi(\lambda_1)g \rangle|})^{-1} \frac{\langle x, \pi(\lambda_2)g \rangle}{|\langle x, \pi(\lambda_2)g \rangle|} = \frac{\sum_{t=0}^2 \omega^t |\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g \rangle|^2}{3|\langle x, \pi(\lambda_1)g \rangle||\langle x, \pi(\lambda_2)g \rangle|}, \quad (4.11)
\]

for \( (\lambda_1, \lambda_2) \in E \). Recall that \( \omega_{\lambda_1, \lambda_2} \) is well defined if and only if \( |\langle x, \pi(\lambda_1)g \rangle| \neq 0 \) and \( |\langle x, \pi(\lambda_2)g \rangle| \neq 0 \). After the relative phases are computed, we use them in the phase propagation process (Algorithm 2) to obtain up-to-a-global-phase frame coefficients of signal \( x \) with respect to \( \Phi_\Lambda \) and recover \( x \) using a dual frame.

Let us further specify \( \Phi_\Lambda \) and \( \Phi_E \) now. For \( \Phi_\Lambda \) we choose the Gabor frame

\[
\Phi_\Lambda = (g, \Lambda) \quad \text{with} \quad \Lambda = F \times \mathbb{Z}_M, \quad F \subset \mathbb{Z}_M,
\]

and \( g \in \mathbb{C}^M \) uniformly distributed on the unit sphere \( S^{M-1} \subset \mathbb{C}^M \). \hfill (4.12)

The integer \( |F| \) is fixed and does not depend on the ambient dimension \( M \). That is, we consider all frequency shifts and only a constant number of time shifts. As equation (2.1) indicates, our measurements with respect to \( \Phi_\Lambda \) are squared magnitudes of masked Fourier transform coefficients with the masks being \( T_{k\bar{g}}, \ k \in F \). That is,

\[
|\langle x, \pi(k, \ell)g \rangle|^2 = |F(x \odot T_{k\bar{g}})(\ell)|^2, \quad \ell \in \mathbb{Z}_M. \quad (4.13)
\]

We choose

\[
\Phi_E = \{ \pi(\lambda_1)g + \omega^{\ell} \pi(\lambda_2)g \}_{(\lambda_1, \lambda_2) \in E, \ t \in \{0,1,2\}} \quad \text{with} \quad \omega = e^{2\pi i/3},
\]

\[
E = \{(k_1, \ell_1), (k_2, \ell_2)\}, \quad \text{s.t.} \quad k_1, k_2 \in F, \quad \ell_2 - \ell_1 \in C \subset \Lambda \times \Lambda, \quad (4.14)
\]

and \( C = D \cup (-D) \setminus \{0\} \subset \mathbb{Z}_M \) with \( 1_D(m) \sim \text{i.i.d.} \ B \left(1, \frac{\log M}{M}\right) \).

In other words, set \( D \) here is a random subset of \( \mathbb{Z}_M \), such that every \( m \in \mathbb{Z}_M \) is chosen to be an element of \( D \) with probability \( \frac{1}{M} \), where \( d > 0 \) is a parameter to be specified later. Then, for \( \lambda_1 = (k_1, \ell_1), \lambda_2 = (k_2, \ell_2) \in \Lambda \), such that \( (\lambda_1, \lambda_2) \in E \),

\[
\pi(\lambda_1)g + \omega^{\ell} \pi(\lambda_2)g = p_{(\ell_2 - \ell_1)k_1k_2}(t) \odot \pi(\lambda_1)g,
\]

which is well defined if and only if \( |\langle x, \pi(\lambda_1)g \rangle| \neq 0 \) and \( |\langle x, \pi(\lambda_2)g \rangle| \neq 0 \).
where the vector $p_{c,k_1,k_2}(t) \in \mathbb{C}^M$ with parameters $c \in C$, $k_1, k_2 \in F$, and $t \in \{0, 1, 2\}$ is defined pointwise by

$$p_{c,k_1,k_2}(t)(m) = 1 + e^{2\pi i \left(\frac{cm + t}{M}\right)} g(m - k_2), \quad m \in \mathbb{Z}_M.$$ 

Therefore, for each fixed set of four parameters $(c, k_1, k_2, t)$, the respective additional measurements are magnitudes of masked Fourier transform coefficients as well. Namely,

$$|\langle x, p_{c,k_1,k_2}(t) \odot \pi(k_1, \ell) \rangle|^2 = |\mathcal{F}(x \odot \bar{p}_{c,k_1,k_2}(t) \odot T_{k_1}(\ell))|^2, \quad \ell \in \mathbb{Z}_M. \quad (4.15)$$

Let us note that the frame $\Phi = \Phi_\Lambda \cup \Phi_E$ constructed in this way consists of $|\Lambda| + 3|E| = |F|M + 3|F|^2|C|M$ vectors. Since $C = D \cup (-D) \setminus \{0\}$ with $1_D(m) \sim \text{i.i.d.} \, B(1, \frac{1}{2\log M})$, we have $|C| = O(\log M)$ with high probability and thus $|\Phi| = O(M \log M)$.

**Remark 4.4.1.** We note that there are some advantages of time-frequency structured frames in comparison to randomly generated frames with independent vectors, such as Gaussian frames. Among such advantages are the following.

1. As equations (4.13) and (4.15) indicate, all required measurements are magnitudes of masked Fourier transform coefficients. Measurements of this type are relevant for many practical applications where phase retrieval problem arises.

2. Measurements and reconstruction in the case of time-frequency structured frames can be implemented using FFT, which allows for a noticeable speed up of measurement and reconstruction processes. For comparison, the computational complexity of the measurement process with random Gaussian frame of cardinality $O(M \log M)$ (as considered, for example, in [17] and [1]) is $O(M^2 \log M)$, and the complexity of measurement with the frame $\Phi$ constructed above is $O(M \log^2 M)$.

3. In the case of random Gaussian frames, we have to use $O(M^2 \log M)$ memory bits to store the measurement matrix, while for our frame $\Phi$ it is enough to store the window $g$ and the set $C$, and the overall amount of memory used is only $M + O(\log M) = O(M)$. In other words, using Gabor frames we reduce the number of random bits required for reconstruction.

### 4.4.2 Reconstruction in the noiseless case

We now describe our polarization based phase retrieval algorithm for time-frequency structured measurements. The reconstruction procedure, which was outlined in Section 4.4.1, is summarized in Algorithm 3.

For $\Lambda$ defined in (4.12) and $E$ defined in (4.14), we consider a graph $G = (\Lambda, E)$, called the graph of measurement. As follows from the description (4.14) of the edge set $E$, a vertex $\lambda = (k, \ell)$ of $G$ is adjacent to any vertex $\lambda' = (k', \ell + c)$ with $c \in C$ and $k' \in F$. Thus, each vertex in $G$ has degree $|F||C|$ and $G$ is regular. Since $0 \notin C$, the graph $G$ has no loops, and since $C = -C$, it is not directed.

We assign to each vertex $\lambda \in \Lambda$ the weight $b_\lambda = |\langle x, \pi(\lambda)g \rangle|^2$ and to each edge $(\lambda_1, \lambda_2) \in E$ the corresponding relative phase $\omega_{\lambda_1,\lambda_2}$ computed in (4.11). The graph
Algorithm 3: Reconstruction in the noiseless case

Input: phaseless measurements \( b \) with respect to \( \Phi_\Lambda \cup \Phi_E \), defined by (4.12) and (4.14); \( F < \mathbb{Z}_M, C \subset \mathbb{Z}_M \), and window \( g \in \mathbb{C}^M \setminus \{0\} \)

Output: \( \hat{x} \in [x] \), initial signal up to a global phase.

1. Construct the graph \( G = (\Lambda, E) \) with \( \Lambda = F \times \mathbb{Z}_M \) and \( E \) as in (4.14);
2. Assign to each \( \lambda \in \Lambda \) the weight \( b_\lambda \);
3. Assign to each edge \( (\lambda_1, \lambda_2) \in E \) the weight \( \omega_{\lambda_1, \lambda_2} \) computed using (4.11);
4. Delete from \( G \) all vertices \( \lambda \) with \( b_\lambda = 0 \) to obtain \( G' = (\Lambda', E') \subset G \);
5. Choose a connected component \( G'' = (\Lambda'', E'') \subset G' \) of the biggest size;
6. Run the phase propagation process (Algorithm 2) to obtain \( c_\Lambda, \lambda \in \Lambda'' \);
7. Reconstruct \( \hat{x} = (\Phi_{\Lambda''} \Phi_{\Lambda''}^*)^{-1} \Phi_{\Lambda''} c \) from \( c = \{ c_\lambda \}_{\lambda \in \Lambda''} \).

of measurements \( G \) is used in the phase propagation process, and the reconstruction guarantees for Algorithm 3 strongly depend on its connectivity properties. Indeed, as discussed in Section 4.2.2, the phase cannot be propagated through vertices with the corresponding zero measurements, and such vertices should be deleted from \( G \). Moreover, phase propagation can be done only on a connected component of the graph \( G' \) obtained after deleting all vertices with zero weights from \( G \). Thus, to be able to reconstruct a signal \( x \) using Algorithm 3, we need to ensure that the size \( |\Lambda''| \) of the biggest connected component \( G'' = (\Lambda'', E'') \) of \( G' \) is sufficiently large, so that \( \Phi_{\Lambda''} = (g, \Lambda'') \) is a frame. Then \( x \) can be recovered from its frame coefficients with respect to \( \Phi_{\Lambda''} \).

As Lemma 2.3.4 shows, graph \( G \) satisfies this property provided its spectral gap is sufficiently big (see Section 2.3 for the definition of the spectral gap of a graph). The spectral gap of \( G = (\Lambda, E) \) can be estimated in terms of the set \( C \). More precisely, it depends on the Fourier bias of \( C \), which is defined in Section 2.2.1, see also [76]. The following lemma, shown in [6], relates the spectral gap of \( G \) and the Fourier bias \( ||C||_u = \max_{m \neq 0} |(F1_C)(m)| \) of \( C \). We include a proof here for completeness, to show that it works also for the graph \( G \) that we constructed above.

Lemma 4.4.2. [6] Consider the graph \( G \) constructed as above. Then the spectral gap of \( G \) satisfies

\[
\text{spg}(G) = 1 - \frac{||C||_u}{|C|}.
\]

Proof. Consider the adjacency matrix \( A = \{ a(k-1)\ell | F| + \ell, (k'-1)|F| + \ell' \}_{(k,\ell), (k',\ell') \in \Lambda} \) of the graph \( G \). For any fixed pair \( k_1, k_2 \in F \) we have

\[
a(k_1-1)\ell_1 | F| + \ell_1, (k_2-1)\ell_2 | F| + \ell_2 = \begin{cases} 
0, & \text{if } \ell_1 - \ell_2 \in C; \\
1, & \text{if } \ell_1 - \ell_2 \notin C.
\end{cases}
\]

It follows that \( A = J \otimes \text{circ}(1_C) \), where \( J \) is the \( |F| \times |F| \) matrix with all entries

---

1An analogous to Lemma 4.4.2 result is shown in [6, Lemma 5] for a specific family of graphs considered there. At the same time, the similarity in the structure of the graph \( G \) we constructed above and the graphs considered in [6] allow us to establish the same result on the spectral gap of \( G \), and the proof of it follows the same steps.
4.4. RECOVERY ALGORITHM FOR TIME-FREQUENCY MEASUREMENTS

Figure 4.3: An example of the graph of measurements \( G \) with \( M = 6 \), \( F = \{0, 3\} \) and \( C = \{2, 3, 4\} \) (left). This graph remains connected after deleting one third of its vertices (middle). After we delete one half of its vertices it has a connected component of size at least 4 (right).

being 1, and \( \text{circ}(1_C) \) is the circulant matrix generated by the vector \( 1_C \), that is

\[
\text{circ}(1_C) = \begin{pmatrix}
1_C(0) & 1_C(M-1) & \cdots & 1_C(1) \\
1_C(1) & 1_C(0) & \cdots & 1_C(2) \\
\vdots & \vdots & \ddots & \vdots \\
1_C(M-1) & 1_C(M-2) & \cdots & 1_C(0)
\end{pmatrix}.
\]

The eigenvalues of the Kronecker product matrix \( A \) are given by the pairwise products of eigenvalues of the factors, namely

\[
\lambda_{j,m}(A) = \lambda_j(J)\lambda_{m}(\text{circ}(1_C)), \ j \in \{1, \ldots, |F|\}, \ m \in \mathbb{Z}_M.
\]

The only nontrivial eigenvalue of \( J \) is \( \lambda_1(J) = |F| \) and \( \lambda_{m}(\text{circ}(1_C)) = \mathcal{F}(1_C)(m) \), for each \( m \in \mathbb{Z}_M \). Thus, for any \( j > 1 \) we have \( \lambda_{j,m} = 0 \) and the nontrivial eigenvalues of \( A \) are given by \( \lambda_{1,m}(A) = |F| \sum_{k \in C} e^{-2\pi i km/M} \). We have

\[
|\lambda_{1,m}(A)| = |F| \left| \sum_{j \in C} e^{-2\pi i jm/M} \right| \leq |F||C|,
\]

with equality if and only if \( m = 0 \). Thus the spectral gap of \( G \) is

\[
\text{spg}(G) = \frac{1}{|F||C|} \left( |F||C| - |F| \max_{m \neq 0} \left( |\mathcal{F}(1_C)(m)| \right) \right) = 1 - \frac{||C||_u}{|C|}.
\]

The following result, relating the Fourier bias of a randomly chosen \( D \subset \mathbb{Z}_M \) and the parameter \( d \) of its distribution, is shown in [6]. The proof is based on a version of the Chernoff bound and Lemma 4.4.2.

**Lemma 4.4.3.** [6] Pick \( d > 36 \) and suppose the entries of the characteristic vector \( 1_D \) of a set \( D \) are independent with distribution \( B \left( 1, \frac{\log M}{M} \right) \). Take \( C = D \cup (-D) \setminus \{0\} \) and construct the graph \( G \) as above. Then, with overwhelming probability,

\[
\text{spg}(G) \geq 1 - \frac{6}{\sqrt{d}}.
\]
Remark 4.4.4. Lemma 4.4.3 shows that if the set $C$ is sufficiently randomized and $|C| = O(\log M)$, then the spectral gap of $G$ is bounded away from zero. Unfortunately, the cardinality of $C$ cannot be reduced beyond the log factor, meaning that, for a graph $G$ constructed as above, $\text{sbg}(G) > \varepsilon$ only if $|C| \geq \frac{\log M}{2+\log(1/\varepsilon)}$, see [6] for the proof.

Now we are ready to establish recovery guarantees for Algorithm 3.

**Theorem 4.4.5.** Let $\Phi_\Lambda$ and $\Phi_F$ be given by (4.12) and (4.14), respectively, with $|F| \geq 12$ and $d \geq 144$. Then every signal $x \in \mathbb{C}^M$ can be reconstructed from $M + 3|F|^2 M |C| = O(M \log M)$ phaseless measurements with respect to the frame $\Phi_\Lambda \cup \Phi_F$ using Algorithm 3.

**Proof.** We begin the reconstruction algorithm with assigning to each vertex $\lambda \in \Lambda$ of the constructed graph $G$ the weight $b_\lambda = |\langle x, \pi(\lambda) g \rangle|^2$ and assigning to each edge $(\lambda_1, \lambda_2) \in E$ of $G$ the relative phase $\omega_{\lambda_1\lambda_2}$ which is computed from the additional edge measurements. Theorem 3.1.2 implies that the frame $\Phi_\Lambda = \{\pi(\lambda) g\}_{\lambda \in \Lambda}$ is full spark with probability one. Thus, for any vector $x \in \mathbb{C}^M$, the number of zero measurements among $\{b_\lambda = |\langle x, \pi(\lambda) g \rangle|^2\}_{\lambda \in \Lambda}$ is at most $M - 1$. In other words, Algorithm 3 deletes at most $M - 1$ vertices from $G$ to obtain $G'$.

Next, $\Phi_\Lambda$ being full spark implies that any its subset $\Phi_{\Lambda'} \subset \Phi_\Lambda$ of size $|\Phi_{\Lambda'}| \geq M$ forms a frame. Thus, to recover $x$, it is enough to know any $M$ of the frame coefficients with respect to $\Phi_{\Lambda'}$.

To show that $G'$ has a connected component $G''$ of size at least $M$, first note that Lemma 4.4.3 ensures that $\text{sbg}(G) \geq 1 - \frac{\varepsilon}{\sqrt{d}} \geq \frac{1}{2}$. Then, applying Lemma 2.3.4 with $n = |\Lambda| = |F| M$ and $\varepsilon = \frac{1}{|F|} \leq \frac{1}{12} \leq \frac{\text{sbg}(G)}{6}$, we obtain that after deleting any $\varepsilon n = M$ vertices from $G$, the largest connected component $G''$ will have at least $\left(1 - \frac{2\varepsilon}{\text{sbg}(G)}\right)n \geq \frac{2}{3}|F| M \geq 8M > M$ vertices.

By running the phase propagation algorithm on the connected component $G'' = (\Lambda'', E'')$, we recover $c_\lambda$, $\lambda \in \Lambda''$, which are, up to a global phase factor $e^{i\theta}$, $\theta \in [0, 2\pi)$, equal to the corresponding frame coefficients of $x$ with respect to $\Phi_{\Lambda''}$, that is,

$$c_\lambda = e^{i\theta} \langle x, \pi(\lambda) g \rangle, \quad \lambda \in \Lambda''.$$ 

Now, using the canonical dual frame, we obtain

$$\hat{x} = (\Phi_{\Lambda''} \Phi_{\Lambda''}^*)^{-1} \Phi_{\Lambda''} c = e^{i\theta} (\Phi_{\Lambda''} \Phi_{\Lambda''}^*)^{-1} \Phi_{\Lambda''} \Phi_{\Lambda''}^* x = e^{i\theta} x.$$

4.5 Robustness of reconstruction in the presence of noise

In many applications, due to imperfection of the measurement process, the measurements are corrupted by noise. Thus, for a phase retrieval algorithm, the important problem is to show that the estimate produced by the algorithm from the noisy measurements is sufficiently close to the initial signal, and the reconstruction error is comparable with the noise level.
In this section our aim is to investigate the behavior of Algorithm 3 constructed in Section 4.4.2 in the case when the available measurements of a signal \( x \in \mathbb{C}^M \) are of the form

\[
b_\lambda = |\langle x, \pi(\lambda)g \rangle|^2 + \nu_\lambda, \quad \lambda \in \Lambda;
b_{\lambda_1,\lambda_2,t} = |\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g \rangle|^2 + \nu_{\lambda_1,\lambda_2,t}, \quad (\lambda_1, \lambda_2) \in E, \quad t \in \{0, 1, 2\}.
\]

Here \( \omega = e^{2\pi i/3} \), \( \Lambda \) and \( E \) are given by (4.12) and (4.14), respectively, and \( \nu_\lambda, \nu_{\lambda_1,\lambda_2,t} \in \mathbb{R} \) are noise terms.

While the phase propagation process used in Algorithm 3 might lead to noise accumulation, we aim to construct a modification of Algorithm 3 which would be robust to the noise in measurements. The analysis of the modified algorithm’s robustness is closely linked to the geometric properties of the measurement frame \( \Phi_\Lambda = (g, \Lambda) \), such as frame order statistics, reflecting “flatness” of the vector of frame coefficients (see Section 3.2), and frame bounds (see Section 3.3). The results from this section can be also found in [65].

### 4.5.1 Modification of the algorithm for the case of noisy measurements

To simplify the discussion, let us assume, without loss of generality, that the signal \( x \) lies on the complex unit sphere \( \mathbb{S}^{M-1} \subset \mathbb{C}^M \). Recall that Algorithm 3 has the following three main steps

1. computation of the relative phases \( \omega_{\lambda_1,\lambda_2} \) using measurements with respect to \( \Phi_E \);
2. finding a connected subgraph \( G'' \) of the graph of measurements \( G = (\Lambda, E) \), such that it does not contain any vertices with corresponding zero measurements;
3. phase propagation on \( G'' \).

To achieve robust reconstruction in the case of noisy measurements (4.16), we analyse each of these steps and track the reconstruction error arising.

**Reliable relative phase estimation**

To compute a relative phase \( \omega_{\lambda_1,\lambda_2} \) between two frame coefficients, we rely on the formula

\[
\omega_{\lambda_1,\lambda_2} = \frac{\langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle}{\langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle} = \frac{\sum_{t=0}^2 \omega^t \langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g \rangle^2}{3 \langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle}.
\]

The calculations include division by the product \( \langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle \) and are therefore very sensitive to perturbations when \( \langle x, \pi(\lambda_1)g \rangle \) or \( \langle x, \pi(\lambda_2)g \rangle \) is small. This means that, while vertices with zero weights provide no relative phase information, vertices with small weights lead to unreliable relative phase estimations. As we show in Section 4.5.2, large measurements \( b_\lambda \) can also prevent robust reconstruction of the initial signal. So, instead of deleting vertices with zero weight, a portion of vertices with weights that are too small or too large should be deleted in the first step of the reconstruction algorithm. To do so, we use Algorithm 4.

In order to reconstruct (an approximation of) the signal \( x \), we need to have enough reliable information on the phases of its frame coefficients. In terms of the
Algorithm 4: Deleting vertices with small and large weights

**Input**: graph \( G = (\Lambda, E) \) with weighted vertex set \( \Lambda \); parameters \( \alpha, \beta \)

**Output**: graph \( G' \) with more “flat” vertex weights

1. for \( i = 0 \) to \((1 - \alpha)|\Lambda|\) do
2. find \( \lambda \in \Lambda \) with the smallest weight \( b_\lambda \);
3. delete the vertex \( \lambda \) from \( G \);
4. end

5. for \( j = 0 \) to \((1 - \beta)|\Lambda|\) do
6. find \( \lambda \in \Lambda \) with the largest weight \( b_\lambda \);
7. delete the vertex \( \lambda \) from \( G \).
8. end

Graph of measurements \( G \), this means that the number of vertices with small or large weights, which are deleted from \( G \) by Algorithm 4, cannot be too large. To have a control on this number, we rely on Theorem 3.2.2, which ensures that, for a fixed \( x \in \mathbb{S}^{M-1} \), with high probability only a small portion of the frame coefficients can be outside of the range \( \left( \frac{c}{\sqrt{M}}, \frac{K}{\sqrt{M}} \right) \), for some suitably chosen constants \( K > c > 0 \).

Angular synchronization using eigenvectors

The phase propagation process, described in Algorithm 2, is an iterative process, meaning that, to compute an estimate of another frame coefficient, we use the information obtained on the previous step, see also Figure 4.2. Thus, noise might accumulate while passing from one vertex of the measurement graph \( G \) to another, and the reconstruction error grows with the number of steps needed to reach all the vertices, that is, with diameter of \( G \).

Thus, to achieve a robust signal reconstruction, we seek a robust and efficient alternative to phase propagation. One such alternative is angular synchronization method, summarised in Algorithm 5 [74, 1]. The main idea behind it is to use all available relative phase information to reduce the noise. More precisely, the angular synchronization approach can be described as follows.

Algorithm 5: Angular synchronization

**Input**: graph \( G'' = (\Lambda'', E'') \) with weighted edges

**Output**: vector \( \tilde{\upsilon} \) approximating the vector of the phases of frame coefficients

1. set \( A \) to be a weighted adjacency matrix defined in (4.17);
2. set \( D = \text{diag}(d_\lambda)_{\lambda \in \Lambda''} \), where \( d_\lambda \) is the degree of the vertex \( \lambda \);
3. compute \( L_1 = I - D^{-1/2}AD^{-1/2} \);
4. compute the eigenvector \( u \) corresponding to the smallest eigenvalue \( \alpha_1 \) of \( L_1 \);
5. set \( \tilde{\upsilon}_\lambda = \frac{u_\lambda}{|u_\lambda|}, \lambda \in \Lambda \).

Let \( G'' = (\Lambda'', E'') \) be a subgraph of \( G' \) and let \( A \) be the weighted adjacency matrix of the graph \( G'' \) given by

\[
A(\lambda_1, \lambda_2) = \begin{cases} \frac{\langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle + \epsilon_{\lambda_1\lambda_2}}{\langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle + \epsilon_{\lambda_1\lambda_2}}, & (\lambda_1, \lambda_2) \in E'', \\ 0, & (\lambda_1, \lambda_2) \notin E''. \end{cases} \quad (4.17)
\]
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where $\lambda_1, \lambda_2 \in \Lambda''$ and $\varepsilon_{\lambda_1\lambda_2} = \frac{1}{3} \sum_{\ell=0}^{2} \omega' \nu_{\lambda_1\lambda_2}\ell$.

Let $\nu_\lambda = \frac{\langle x, \pi(\lambda)g \rangle}{\langle x, \pi(\lambda)g \rangle}$, $\lambda \in \Lambda''$, be the unknown true phases of the frame coefficients. Then, for $(\lambda_1, \lambda_2) \in E$, $A(\lambda_1, \lambda_2)$ can be considered as an approximation of the relative phase $\omega_{\lambda_1\lambda_2} = \nu_{\lambda_1}^{-1} \nu_{\lambda_2}$. Consider the vector $\tilde{v} = \{\tilde{v}_\lambda\}_{\lambda \in \Lambda}$ given by

$$\tilde{v} = \arg\min_{|u_\lambda| = 1, \lambda \in \Lambda'\prime} \sum_{(\lambda_1, \lambda_2) \in E''} |u_{\lambda_2} - A(\lambda_1, \lambda_2)u_{\lambda_1}|^2 = \arg\min_{|u_\lambda| = 1, \lambda \in \Lambda''} u^*(D - \tilde{A})u,$$

(4.18)

where $D$ is the diagonal matrix of vertex degrees and $\tilde{A}$ is a componentwise conjugate of $A$. The vector $\tilde{v}$ is an approximation of $v$.

Since we assume $|u_\lambda| = 1$, $\lambda \in \Lambda''$, the quantity $u^*Du = \sum_{\lambda \in \Lambda''} d_\lambda |u_\lambda|^2 = \sum_{\lambda \in \Lambda''} d_\lambda$ does not vary with $u$. Thus, (4.18) is equivalent to minimizing

$$\frac{u^*(D - \tilde{A})u}{u^*Du} = \frac{(D^{1/2}u)^*(I - D^{-1/2}\tilde{A}D^{-1/2})(D^{1/2}u)}{||D^{1/2}u||^2_2}.$$

This quantity is larger than or equal to the smallest eigenvalue $\alpha_1$ of the connected Laplacian matrix $L_1 = I - D^{-1/2}\tilde{A}D^{-1/2}$, and the equality is achieved if $D^{1/2}u = u(\alpha_1)$, where $u(\alpha_1)$ is a corresponding eigenvector.

Note that in the case when $G''$ is connected and no noise is present in the measurements, $\tilde{v} = D^{-1/2}u(\alpha_1)$ (after a suitable renormalization) satisfies the condition $|\tilde{v}_\lambda| = 1, \lambda \in \Lambda''$, and is the unique minimizer of (4.18). In the case of noisy measurements, we use the same approach by relaxing the nonconvex condition $|u_\lambda| = 1, \lambda \in \Lambda''$. The following result obtained in [1, 7] shows robustness of the Algorithm 5.

**Theorem 4.5.1.** [1] Consider a graph $G'' = (\Lambda'', E'')$ with spectral gap $\tau > 0$, and define $||\theta||_T = \min_{k \in \mathbb{Z}} |\theta - 2\pi k|$ for all angles $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Then, given the weighted adjacency matrix $A$ as in (4.17), Algorithm 5 outputs $\hat{v} \in \mathbb{C}^{|\Lambda''|}$ with unit-modulus entries, such that for some phase $\theta \in [0, 2\pi),$

$$\sum_{\lambda \in \Lambda''} ||\arg(\tilde{v}_\lambda) - \arg(\langle x, \pi(\lambda)g \rangle) - \theta||^2_2 \leq \frac{C||\varepsilon||^2}{\tau^2 P^2},$$

where $P = \min_{(\lambda_1, \lambda_2) \in E''} |\langle x, \pi(\lambda_1)g \rangle \langle x, \pi(\lambda_2)g \rangle + \varepsilon_{\lambda_1\lambda_2}|$ and $C$ is a universal constant.

**Finding subgraphs with big spectral gap**

As Theorem 4.5.1 shows, the accuracy of Algorithm 5 depends on the spectral gap of the graph $G''$. To find a subgraph $G'' \subset G'$ with spectral gap bounded away from zero, we use the spectral clustering algorithm [62, 1].

The spectral clustering approach is based on the following idea. Let us consider a random walk on a graph $G'$. It is intuitively clear that, if $G'$ has a well-connected subgraph that has only few connections to the rest of the vertices, then after a while the random walk would be trapped in this subgraph, meaning that it would stay inside the subgraph with high probability. This idea is formalized in [58, 61]. In particular, it is known that such subgraphs can be identified using the second eigenvector of the corresponding stochastic matrix.

To show that the graph $G'' = (\Lambda'', E'')$ obtained after applying Algorithm 6 is big enough, that is, $|\Lambda''| \geq M$, we use the following result. Its proof is based on the Cheeger inequality for the graph connected Laplacian [21].
Algorithm 6: Spectral clustering

Input : graph $G' = (V', E')$, parameter $\tau$
Output: graph $G''$ with spectral gap at least $\tau$
1 set $G'' = G'$;
2 while $\text{spg}(G'') < \tau$ do
3 set $D = \text{diag}(d_1, \ldots, d_n)$, where $d_i$ is the degree of the $i$th vertex;
4 set $A$ to be the adjacency matrix of $G''$;
5 compute the connected Laplacian $L = I - D^{-1/2}AD^{-1/2}$;
6 let $u$ be an eigenvector of $L$ with the second smallest eigenvalue $\alpha_2$;
7 for $i = 1$ to $|V|/2$ do
8 let $S_i$ be a set of vertices corresponding to $i$ smallest entries of vector $D^{-1/2}u$;
9 set $h_i = \frac{|E(S_i, S_i^c)|}{\min\{\sum_{v \in S_i} \deg v, \sum_{v \in S_i^c} \deg v\}}$;
10 end
11 set $G'' = G'' \setminus S$, where $S = S_i$ with the minimal corresponding $h_i$.
end

Theorem 4.5.2. [1] For any $1 > p \geq q \geq \frac{2}{3}$, consider a regular graph $G = (V, E)$ with spectral gap $\text{spg}(G) > g(p, q) = 1 - 2(q(1-q) - (1-p))$. Then, after Algorithm 4 removes at most $(1-p)|V|$ vertices from $G$, Algorithm 6 outputs a subgraph with at least $q|V|$ vertices and spectral gap at least $\tau = \frac{1}{8}(\text{spg}(G) - g(p, q))^2$.

4.5.2 Robustness guarantees

We now incorporate the modifications discussed in Section 4.5.1 into the reconstruction process and obtain the improved phase retrieval method summarized in Algorithm 7. The following theorem shows the robustness of the obtained method in the presence of noise.

Algorithm 7: Reconstruction in the noisy case

Input : phaseless noisy measurements $b$ w.r.t. $\Phi_\Lambda \cup \Phi_E$, given by (4.16); $F < \mathbb{Z}_M$, set $C \subset \mathbb{Z}_M$, and window $g \in \mathbb{C}^M$; parameters $\alpha, \beta, \tau$
Output: approximation $\hat{x}$ of the initial signal $x$ (up to a global phase)
1 construct the graph $G = (\Lambda, E)$ with $\Lambda = F \times \mathbb{Z}_M$ and $E$, given by (4.14);
2 assign to each $\lambda \in \Lambda$ the weight $b_\lambda$;
3 assign to each edge $(\lambda_1, \lambda_2) \in E$ the weight $A_{\lambda_1, \lambda_2}$, given by (4.17);
4 use Algorithm 4 with parameters $\alpha, \beta$ to obtain $G' = (\Lambda', E') \subset G$;
5 use Algorithm 6 with parameter $\tau$ to find a subgraph $G'' = (\Lambda'', E'') \subset G'$ with $\text{spg}(G'') \geq \tau$;
6 use Algorithm 5 to obtain approximate phases $\{u_\lambda\}_{\lambda \in \Lambda''}$ of the frame coefficients of $x$ w.r.t. $\Phi_{\Lambda''}$;
7 set $c_\lambda = u_\lambda \sqrt{b_\lambda}$, $\lambda \in \Lambda''$;
8 reconstruct $\hat{x} = (\Phi_{\Lambda''} \Phi_{\Lambda''}^*)^{-1} \Phi_{\Lambda''} c$ from $c = \{c_\lambda\}_{\lambda \in \Lambda''}$.
Theorem 4.5.3. Fix $x \in \mathbb{C}^M$ and consider the measurement procedure (4.16) with $|F|$ and $d$ sufficiently large. If the noise vector satisfies $\frac{||\nu||_2}{||\nu||_2} \leq \frac{C_1}{\sqrt{d}}$ for some $C_1$ small enough, then there exists a constant $C''$ such that with overwhelming probability the estimate $\hat{x}$ produced by Algorithm 7 satisfies

$$\min_{\theta \in (0,2\pi)} \left| \left| \hat{x} - e^{i\theta} x \right| \right|_2^2 \leq \frac{C'' \sqrt{M} ||\nu||_2}{\Delta},$$

where $\Delta = \min_{\lambda \in \Lambda : |\lambda| \geq 2/3|\Lambda|} \sigma^2_{\min}(\Phi^*_\lambda)$.

Proof. Without loss of generality we can assume that $\left| \left| x \right| \right|_2 = 1$. As follows from Lemma 4.4.3 and the construction of the graph of measurements $G = (\Lambda, E)$ given by (4.12) and (4.14), with overwhelming probability $\text{spg}(G) \geq 1 - \frac{6}{\sqrt{d}}$. Let us fix parameters $\tau_0 > 0$, $\alpha, \beta \in (0,1)$ and apply Theorem 4.5.2 with $p = \alpha + \beta$ and $q$ satisfying $g(p,q) = 1 - 2(q(1-q) - (1-p)) = 1 - \frac{6}{\sqrt{d}} - \tau_0 < \text{spg}(G)$.

After Algorithm 4 deletes $(1-p)|\Lambda| = (1-\alpha - \beta)|\Lambda|$ vertices with the smallest and the largest corresponding measurements, we apply Algorithm 6 with parameter $\tau = \frac{1}{8}(\text{spg}(G) - g(p,q))^2 \geq \frac{\alpha^2}{8}$ to obtain a graph $G'' = (\Lambda'', E'')$. Then it follows from Theorem 4.5.2 that $|\Lambda''| \geq q|\Lambda|$ and $\text{spg}(G'') \geq \frac{7d}{8}$.

Let us estimate $q$ now. Since we have $1 - 2(q(1-q) - (1-p)) = 1 - \frac{6}{\sqrt{d}} - \tau_0$, it follows that $q(1-q) = \frac{3}{\sqrt{d}} + (1-\alpha-\beta) + \frac{\tau_0}{2}$. Let us choose parameters $\tau_0, \alpha, \beta$, and $d$ so that $\frac{3}{\sqrt{d}} + (1-\alpha-\beta) + \frac{\tau_0}{2} = A \leq \frac{2}{3}$. Then $1 \geq q = 1 + \sqrt{1 - 4A} \geq 1 + \sqrt{1/9} = \frac{2}{3}$. This ensures that $q \in (\frac{2}{3}, 1)$.

Now, after we found a subgraph $G'' = (\Lambda'', E'')$ with $\text{spg}(G'') \geq \frac{7d}{8}$ and $|\Lambda''| \geq \frac{2}{3}|\Lambda|$, we apply Algorithm 5 to compute estimates $\{u_\lambda\}_{\lambda \in \Lambda''}$ of the phases of frame coefficients $\{x, \pi(\lambda)g\}_\lambda$. Theorem 4.5.1 implies that there exist a universal constant $C > 0$ and phase $\theta \in [0,2\pi)$, such that

$$\sum_{\lambda \in \Lambda''} \left| \arg(u_\lambda) - \arg(\langle x, \pi(\lambda)g \rangle) - \theta \right|^2 \leq \frac{64C||\nu||_2^2}{\tau_0^2 P^2}, \quad (4.19)$$

where $\varepsilon_{\lambda_1\lambda_2} = \frac{1}{3} \sum_{t=0}^2 \omega^t \nu_{\lambda_1\lambda_2}$, and $P = \min_{(\lambda_1, \lambda_2) \in E''} |\langle x, \pi(\lambda_1)g \rangle| |\langle x, \pi(\lambda_2)g \rangle| + \varepsilon_{\lambda_1\lambda_2}$.

Since $||\nu||_2 \leq \frac{C}{\sqrt{d}}$, we also have $|\nu_{\lambda_1\lambda_2}| \leq \frac{C_0}{\sqrt{d}}$ for all $(\lambda_1, \lambda_2) \in E''$ and $t \in \{0, 1, 2\}$, and thus $|\varepsilon_{\lambda_1\lambda_2}| \leq \frac{C_0}{\sqrt{d}}$. By the Cauchy-Schwarz inequality we have

$$||\varepsilon||_2^2 = \sum_{(\lambda_1, \lambda_2) \in E''} |\varepsilon_{\lambda_1\lambda_2}|^2 \leq \frac{1}{9} \sum_{(\lambda_1, \lambda_2) \in E''} \left( \sum_{t=0}^2 |\omega^t|^2 \right) \left( \sum_{t=0}^2 |\nu_{\lambda_1\lambda_2}|^2 \right) \leq \frac{1}{3} ||\nu||_2^2.$$

Theorem 3.2.2(ii) implies that, for any $k, c > 0$ and $\varepsilon = c^2$,

$$\left| \{ \lambda \in \Lambda, \text{s.t.} \left| \langle x, \pi(\lambda)g \rangle \right| < \frac{c}{\sqrt{M}} \} \right| < |\Lambda|\left( \varepsilon + k\sqrt{2}\varepsilon \right)$$

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with probability at least $1 - 1/k^2$. Then, since $|\nu_\lambda| \leq \frac{C_i}{M}$,
\[
\left\{ \lambda \in \Lambda, \text { s.t. } |\langle x, \pi(\lambda)g \rangle|^2 + \nu_\lambda < \frac{\epsilon^2}{M} - \frac{C_1}{M} \right\} \subseteq \left\{ \lambda \in \Lambda, \text { s.t. } |\langle x, \pi(\lambda)g \rangle|^2 < \frac{\epsilon^2}{M} \right\},
\]
that is, with the same probability
\[
\left\{ \lambda \in \Lambda, \text { s.t. } |\langle x, \pi(\lambda)g \rangle|^2 + \nu_\lambda < \frac{\epsilon^2 - C_1}{M} \right\} \subseteq |\Lambda| \epsilon k \sqrt{2}\epsilon).
\]

We set $\alpha = 1 - (\epsilon + k \sqrt{2}\epsilon)$, so that Algorithm 4 deletes $|\Lambda| \epsilon k \sqrt{2}\epsilon$ vertices with the smallest corresponding measurements. Then, provided $\epsilon^2 > 2C_1$, with probability at least $1 - 1/k^2$ for the remaining coefficients we have $|\langle x, \pi(\lambda)g \rangle| \geq \frac{\epsilon}{\sqrt{M}}$, for $\epsilon = \sqrt{\epsilon^2 - 2C_1} > 0$.

Similarly, for any $K, k > 0$ and $\eta = \frac{8}{\pi} e^{-K^2} + k \sqrt{2}\eta$ and $\beta = 1 - (\eta + k \sqrt{2}\eta)$, so that Algorithm 4 deletes $|\Lambda| \epsilon k \sqrt{2}\epsilon$ vertices with the largest corresponding measurements. Then Theorem 3.2.2(ii) implies that with probability at least $1 - 1/k^2$ for the remaining vertices we have $b_\lambda \leq \frac{K + C_1}{M} = \frac{K}{M}$, and thus $|\langle x, \pi(\lambda)g \rangle| \leq \frac{\sqrt{K + 2C_2}}{\sqrt{M}}$.

Thus, after we applied Algorithms 4 and 6, with probability at least $1 - k^2$ we have $\frac{\epsilon}{\sqrt{M}} \leq |\langle x, \pi(\lambda)g \rangle| \leq \frac{\sqrt{K + 2C_2}}{\sqrt{M}}$ for all $\lambda \in \Lambda''$. This implies that for a suitably chosen constant $\tilde{C} > 0$
\[
P = \min_{(\lambda_1, \lambda_2) \in E''} |\langle x, \pi(\lambda_1)g \rangle| \langle x, \pi(\lambda_2)g \rangle + \epsilon_{\lambda_1, \lambda_2} \geq \frac{\epsilon^2}{M} - \frac{C_1}{M} \geq \frac{\tilde{C}}{M}.
\]

Substituting the obtained bounds on $||\varepsilon||^2_2$ and $P$ into (4.19), we obtain
\[
\sum_{\lambda \in V''} ||\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta||^2_2 \leq \frac{64C||\nu||^2_2 M}{3\tau_0^4 C^2}.
\]

For every $\lambda \in \Lambda''$, we denote the obtained estimate of the corresponding frame coefficient by $c_\lambda = u_\lambda \sqrt{\langle x, \pi(\lambda)g \rangle^2 + \nu_\lambda}$. We also set $\delta_\lambda = c_\lambda - \epsilon e^{i \theta} \langle x, \pi(\lambda)g \rangle$ and $\xi_\lambda = \sqrt{b_\lambda} - |\langle x, \pi(\lambda)\rangle|$. Then
\[
|\delta_\lambda| = \left| \sqrt{b_\lambda} e^{i \arg(u_\lambda)} - \sqrt{b_\lambda} e^{i (\theta + \arg((x, \pi(\lambda)g)))} + \xi_\lambda e^{i (\theta + \arg((x, \pi(\lambda)g)))} \right|
\leq \sqrt{b_\lambda} \left| e^{i (\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta)} - 1 \right| + |\xi_\lambda|
\leq \sqrt{b_\lambda} ||\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta||_2 + |\xi_\lambda|.
\]

Furthermore, since $|\delta_\lambda|^2 \leq 2 b_\lambda ||\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta||^2_2 + 2 |\xi_\lambda|^2$, it follows
\[
||\delta||^2_2 \leq 2 \sum_{\lambda \in \Lambda''} b_\lambda ||\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta||^2_2 + 2 \sum_{\lambda \in \Lambda''} |\xi_\lambda|^2.
\]

Using the fact that, for any $a, b \in \mathbb{R}$, $(a - b)^2 \leq |a^2 - b^2|$, we obtain $\xi_\lambda^2 = \left( \sqrt{b_\lambda} - |\langle x, \pi(\lambda)\rangle| \right)^2 \leq |\nu_\lambda|$. And, since $b_\lambda \leq \frac{K}{M}$ and $||.||_1 \leq \sqrt{\Lambda''}||.||_2$, we get
\[
||\delta||^2_2 \leq \frac{2K}{M} \sum_{\lambda \in V''} ||\arg(u_\lambda) - \arg((x, \pi(\lambda)g)) - \theta||^2_2 + 2 ||\nu||_1
\leq \frac{128C K ||\nu||_2^2 M}{3\tau_0^4 C^2} + 2 \sqrt{|F|M ||\nu||_2} \leq C'' \sqrt{|M||\nu||_2},
\]

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For our numerical simulations, we consider noisy phaseless measurements \(D\) be an element of \(\mathbb{C}^m\). In other words, \(D\) is constructed at random, so that every \(m \in \mathbb{Z}_M\) is chosen to be an element of \(D\) independently with probability \(\frac{d \log M}{M}\), for some parameter \(d > 0\).

The measurement frame \(\Phi\) and the set of vectors for additional measurements \(\Phi_E\) are then given by

\[
\Phi_E = \{\pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\}_{\lambda_1, \lambda_2 \in E, \ t \in \{0, 1, 2\}} \quad \text{with} \quad \omega = e^{2mi/3}.
\]

For our numerical simulations, we consider noisy phaseless measurements

\[
b_\lambda = |\langle x, \pi(\lambda)g \rangle|^2 + \nu_\lambda, \quad \lambda \in \Lambda;
\]

\[
b_{\lambda_1 \lambda_2 t} = |\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g \rangle|^2 + \nu_{\lambda_1 \lambda_2 t}, \quad (\lambda_1, \lambda_2) \in E, \ t \in \{0, 1, 2\},
\]

where \(\nu_\lambda, \nu_{\lambda_1 \lambda_2 t} \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)\) are independent normally distributed additive noise components. Theorem 4.5.3 then gives the following bound on the reconstruction error of Algorithm 7

\[
||\hat{x} - e^{i\theta}x||_2^2 \leq C\sqrt{M}||\nu||_2,
\]

(4.21)

where the constant \(C\) depends on the spectral gap of the graph of measurements \(G\) and on the parameter \(\Delta = \min_{\lambda \in \Lambda, |\lambda'| \geq 2/3|\lambda|} \sigma^2_{\min}(\Phi_{\lambda'})\). We discuss parameter \(\Delta\) in Section 3.3.1, see Figure 3.5 and Conjecture 3.3.12.
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Figure 4.4: Dependence of the reconstruction error of Algorithm 7 (left) and the error to noise ratio (right) on the ambient dimension $M$. Here the noise vector is random, such that it has independent normally distributed entries with variance $\sigma = 10^{-3}$. The black dashed lines on the plots show the average (over several simulations with different noise realizations) values of the reconstruction error and the error to noise ratio, respectively. These numerical results suggest that the error to noise ratio does not depend on the signal dimension and is bounded above by a numerical constant.

To illustrate Theorem 4.5.3, we consider two sets of simulations. For the first one, we let the dimension of the signal vary and explore the reconstruction error of the algorithm for a random normally distributed noise vector with independent entries and fixed variance. On Figure 4.4 we show the obtained results, which suggest that the error to noise ratio does not depend on the signal dimension, unlike the bound (4.21) obtained in Theorem 4.5.3. In fact, the ratio between the reconstruction error and the norm of the noise vector appears to be bounded above by a numerical constant close to 3.

For the second set of simulation, we explore the dependence of the reconstruction error and the error to noise ratio on the noise variance for a fixed signal dimension. The obtained results, shown on Figure 4.5, illustrate that the reconstruction error grows linearly with the magnitude of noise.

On both Figures 4.4 and 4.5, we show the average values of the reconstruction error and the error to noise ratio (over several simulations with different noise realizations) using black dashed lines. In other words, the black dashed lines on the plots show an approximation of the expected values of the corresponding quantities. We note that, on both figures, the average error to noise ratio appears to be smaller than 1, which means that noise reduction takes place during signal reconstruction. This can be explained in the following way. Assuming that the graph of measurements $G$ is sufficiently well connected, that is, $\text{spg}(G)$ is sufficiently big, the phase of a frame coefficient can be propagated to the corresponding vertex using various different paths. In Algorithm 7, we use the angular synchronization algorithm (Algorithm 5), which utilizes relative phase information coming to a vertex $\lambda \in \Lambda$ from all edges $(\lambda, \lambda') \in E$ incident to $\lambda$. Since in the simulations we considered noise with independent entries and zero mean, it tends to cancel itself at a vertex.

The reason why the plots on Figures 4.4 and 4.5 look quite spiky is that different realizations of the random graph of measurements $G$ are used for the simulations. As we mentioned before, the reconstruction error bound (4.21) of Algorithm 7 depends
4.5. ROBUSTNESS OF RECONSTRUCTION IN THE PRESENCE OF NOISE

Figure 4.5: Dependence of the reconstruction error of Algorithm 7 (left) and the error to noise ratio (right) on the variance $\sigma$ of the entries $\nu_\lambda, \nu_{\lambda_1\lambda_2\ell}$ of the noise vector $\nu$, which are selected independently from the normal distribution $\mathcal{N}(0, \sigma)$. The ambient dimension here is $M = 100$. The black dashed lines on the plots show the average (over several simulations with different noise realizations) values of the reconstruction error and the error to noise ratio, respectively. These numerical results suggest that the reconstruction error grows linearly with the magnitude of noise.

on the spectral gap of $G$, which might differ from one realization to another. In particular, the bigger the cardinality of the random set $C$ is, the better are the connectivity properties of $G$, and the smaller is the reconstruction error.

The cardinality of the set $C$ and, thus, the reconstruction error of the algorithm depend on the parameter $d$, as formula (4.20) shows. More precisely, it follows from Lemma 4.4.3 that, provided $d > 36$,

$$\text{spg}(G) \geq 1 - \frac{6}{\sqrt{d}}$$

with overwhelming probability. We now investigate the dependence of the reconstruction error of the algorithm on the parameter $d$ numerically.

Numerical results presented on Figure 4.6 show the dependence of the ratio between the reconstruction error of Algorithm 7 and the norm of the noise vector on the parameter $d$ (vertical axis) for the varying ambient dimension (horizontal axis). One can see that starting approximately at $d = 3$, that is, much earlier than the value $d = 144$, predicted by Theorem 4.5.3, this ratio does not exceed 4.

The numerical results presented in this section suggest that the theoretical bound (4.21) on the reconstruction error of the proposed phase retrieval algorithm can be further improved by the factor of $\sqrt{M}$. Thus, one of the important tasks for the future research is to understand the gap between theoretically predicted robustness guarantees and results obtained numerically. We hope that further study of properties of Gabor frames with random windows would allow us not only to remove the factor of $\sqrt{M}$ from the reconstruction error bound (4.21), but also to prove the following conjecture, which states the uniform robustness guarantees for Algorithm 7.

**Conjecture 4.5.4.** Consider the measurement procedure (4.16) with $|F|$ and $d$ sufficiently large. If the noise vector satisfies $\frac{|\nu||\nu|}{||x||^2} \leq \frac{C_1}{M}$ for some $C_1$ small enough, then there exists a numerical constant $C > 0$ such that with overwhelming probability, for
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Figure 4.6: Dependence of the reconstruction error to noise ratio of Algorithm 7 on the parameter $d$ in (4.20) for various dimensions. Parameter $d$ controls the connectivity properties of the graph of measurements $G$. These numerical results suggest that, starting at approximately $d = 3$, the error to noise ratio does not exceed 4, and also does not depend on the ambient dimension $M$. This observation allows to reduce the multiplicative constant in the number of measurements required for the reconstruction.

every $x \in \mathbb{C}^M$, the estimate $\tilde{x}$ produced by Algorithm 7 satisfies

$$\min_{\theta \in [0,2\pi)} ||\tilde{x} - e^{i\theta}x||^2 \leq C||\nu||_2.$$  

We note that one of the main ingredients of the proof of Theorem 4.5.3 is Theorem 3.2.2, which gives bounds on the frame order statistics of a Gabor frame with a random window (see Section 3.2). Similarly, the main missing ingredient of the proof of Conjecture 4.5.4 is a uniform version of Theorem 3.2.2. In particular, should Conjecture 3.2.9 be true, Conjecture 4.5.4 would follow.
Appendix A

Frame order statistics for random frames

Here we consider a class of random frames with independent frame vectors satisfying certain moment assumptions. We note, that the class of considered random frames is quite large, and, in particular, it includes Gaussian random frames with independent frame vectors, considered in [1], and also subgaussian random frames (see Section 2.2 for the definition).

For the considered class of frames, we provide bounds on the α-smallest uniform frame order statistics (Theorem A.2.1) and on the β-largest uniform frame order statistics (Theorem A.3.1). Roughly speaking, these results show that, with overwhelming probability, the vector of frame coefficients is essentially “flat” for all $x \in \mathbb{S}^{M-1}$, meaning that all except a small portion of frame coefficients are (in modulus) inside the interval $(c \sqrt{M}, K \sqrt{M})$, for some suitably chosen $K > c > 0$. The reader is referred to Section 3.2, Definition 3.2.4, for the precise definition of the uniform frame order statistics.

A.1 Littlewood-Offord problem and small ball probability

Consider a random sum

$$S(a, \xi) = \sum_{k=1}^{M} a_k \xi_k,$$

where $\xi_1, \ldots, \xi_M$ are independent identically distributed random variables and $a = (a_1, \ldots, a_M)$ is a vector with real coefficients.

The large deviation theory establishes that $S$ is nicely concentrated around its mean. On the other hand, by the central limit theorem, one cannot expect tighter concentration than the one achieved by appropriately scaled Gaussian random variable. However, rigorous anti-concentration estimates are hard to prove. The Littlewood-Offord problem, first proposed in [52], asks to estimate the small ball probability

$$p_\varepsilon(a) = \sup_{v \in \mathbb{R}} \mathbb{P}(|S(a, \xi) - v| \leq \varepsilon).$$

A small value of $p_\varepsilon(a)$ means that the random sum $S(a, \xi)$ is well spread, in the sense that it does not concentrate in a small neighborhood of any $v \in \mathbb{R}$. 

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Definition A.1.1.

(i) (Essential LCD). Let \( \rho \in (0, 1) \), \( \kappa \geq 0 \), and \( a \in \mathbb{R}^M \). Then

\[
D_{\rho, \kappa}(a) = \inf \{ t > 0 : \exists I \subset \{1, \ldots, M\}, |I| = \kappa, \text{ s.t. dist}((ta)_i, \mathbb{Z} \setminus \{0\}) \leq \rho, \forall i \in I^c \}
\]

is called the essential least common denominator (or essential LCD) of the vector \( a \in \mathbb{R}^M \).

(ii) (Spread part). Let \( 0 < K_1 < K_2 \) be fixed. For a vector \( x \in \mathbb{R}^M \), we consider

\[
\sigma(x) = \{ k \in \{1, \ldots, M\}, K_1 \leq |M^{1/2}x(k)| \leq K_2 \}.
\]

Then, if \( \sigma(x) \neq \emptyset \), we define the spread part of \( x \) as

\[
\hat{x} = (M^{1/2}x(k))_{k \in \sigma(x)}.
\]

If \( \sigma(x) = \emptyset \), the spread part of \( x \) is not defined.

In words, essential LCD of a vector \( a \in \mathbb{R}^M \) is the infimum over all \( t > 0 \), such that all except \( \kappa \) coordinates of the vector \( ta \) are of distance at most \( \rho \) from nonzero integers.

Example A.1.2. Consider \( a \in (\mathbb{Z} \setminus \{0\})^M \). Even for this very simple case, it is a non-trivial task to compute \( D_{\rho, \kappa}(a) \). Here we make some useful observations.

1. Let us define \( \gcd(a, \kappa) = \max_{I \subset \{1, \ldots, M\}, |I| = \kappa} \gcd\{a_i\}_{i \in I^c} \). Then, clearly,

\[
D_{\rho, \kappa}(a) \leq \frac{1}{\gcd(a, \kappa)},
\]

since all except at most \( \kappa \) coefficients of \( \frac{1}{\gcd(a, \kappa)}a \) are integers. One can also show a stronger bound

\[
D_{\rho, \kappa}(a) \leq \frac{1 - \rho/a_{\text{max}}}{\gcd(a, \kappa)},
\]

where \( a_{\text{max}} = \max_{i \in I_{\text{max}}} \frac{|a_i|}{\gcd(a, \kappa)} \) with \( I_{\text{max}} = \arg \max_{I \subset \{1, \ldots, M\}, |I| = \kappa} \gcd\{a_i\}_{i \in I^c} \).

Indeed, for each \( i \in I_{\text{max}} \), we have

\[
\text{dist}\left( a_i \frac{1 - \rho/a_{\text{max}}}{\gcd(a, \kappa)}, \mathbb{Z} \setminus \{0\} \right) \leq \text{dist}\left( \frac{a_i}{\gcd(a, \kappa)} - \frac{\rho a_i}{a_{\text{max}} \gcd(a, \kappa)} \frac{\rho a_i}{\gcd(a, \kappa)} \right) = \frac{\rho a_i}{a_{\text{max}} \gcd(a, \kappa)} \leq \rho.
\]

However, this bound is not exact. Consider for example \( a = (8, 13, 15, 20, 22), \kappa = 1, \) and \( \rho = \frac{1}{7} \). Then the bound above gives us \( D_{\rho, \kappa}(a) \leq \frac{1 - \rho/a_{\text{max}}}{\gcd(a, \kappa)} = \frac{129}{130} \). But we clearly have \( D_{\rho, \kappa}(a) \leq \frac{1}{7} \), since \( \frac{1}{7} = (1\frac{1}{7}, 1\frac{2}{7}, 2\frac{1}{7}, 2\frac{2}{7}, 2\frac{3}{7}) \).

2. There is also a lower bound

\[
D_{\rho, \kappa}(a) \geq \frac{1 - \rho}{a_{\text{min}}},
\]

where \( a_{\text{min}} = \max_{I \subset \{1, \ldots, M\}, |I| = \kappa} \min_{i \in I^c} |a_i| \), that is, \( a_{\text{min}} \) is the \((\kappa + 1)\)th smallest in modulus entry of \( a \).
Indeed, if $t < \frac{1-\varrho}{a_{\min}}$, then for at least $\kappa + 1$ coordinates $a_k$ we have
\[
t|a_k| < \frac{(1-\varrho)|a_k|}{a_{\min}} \leq 1-\varrho,
\]
and thus $\text{dist}(ta_k, \mathbb{Z} \setminus \{0\}) > \varrho$. Note, that this bound is true not only for vectors with integer coefficients, but also for a general vector $a \in \mathbb{R}^M$.

This bound is exact for some vectors $a$. Consider for example $a = (1, 3, 5, 10, 19)$, $\kappa = 1$, and $\varrho = \frac{1}{3}$. Then $a_{\min} = 3$, and $\frac{1-\varrho}{a_{\min}}a = \left(\frac{2}{9}, \frac{2}{3}, 1, \frac{1}{9}, 2\frac{2}{9}, 4\frac{2}{9}\right)$, thus $D_{\frac{1}{3},1}(a) = \frac{2}{9}$.

This bound is also attained for any $a$ and $\kappa$ whenever $\varrho \geq \frac{1}{2}$. However, one can check that for the vector $a = (8, 13, 15, 20, 22)$, $D_{\frac{1}{3},1}(a) > \frac{6}{91} = \frac{1-\varrho}{a_{\min}}$.

Rudelson and Vershynin proved the following result that gives an essentially sharp bound on the small ball probability $p_\varepsilon(a)$ in terms of the essential LCD of the spread part of the corresponding vector $a \in \mathbb{R}^M$ [69].

**Theorem A.1.3.** [69] (Small ball probability). Let $\xi_1, \ldots, \xi_M$ be independent identically distributed centred random variables with variance at least 1 and third moment bounded by $B$. Let $a \in \mathbb{R}^M$ be such that the spread part $\hat{a}$ is well defined for some $K_1, K_2 > 0$, and let $\varrho \in (0, 1)$, $\beta \in (0, \frac{1}{2})$. Then, for any $\varepsilon > 0$,
\[
p_\varepsilon(a) \leq \frac{C}{\sqrt{\beta}} \left( \varepsilon + \frac{1}{\sqrt{MD_{\varrho,\beta M}(\hat{a})}} \right) + Ce^{-c\varrho^2\beta M},
\]
where $C, c > 0$ depend only on $B$, $K_1$, and $K_2$.

**Remark A.1.4.** Theorem A.1.3 is proven in [69] for $a \in \mathbb{R}^M$, as it is stated here. However, the proof can be easily generalize to the complex case, when $a \in \mathbb{C}^M$ and $\xi_i$, $i \in \{1, \ldots, M\}$, are complex valued random variables, see for example [63]. In this case Theorem A.1.3 still holds if we replace $D_{\varrho,\beta M}(\hat{a})$ with $D_{\varrho,\beta M}(|\hat{a}|)$, where $|\cdot|$ denotes the pointwise absolute value. In this manuscript, we use both real and complex case versions of Theorem A.1.3.

### A.2 Small frame coefficients

In this section, we use Theorem A.1.3 to obtain a bound on the $\alpha$-smallest uniform frame order statistics for a random frame with independent frame vectors under certain fourth moment assumptions. We note that Theorem A.2.1 proven below is a stronger version of [1, Lemma 6.9], as it covers a wider class of random frames with independent frame vectors. This result has been also shown in the Master’s thesis of the author of this manuscript [72].

**Theorem A.2.1.** Let the frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ with $M$ large enough be such that $\varphi_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, are independent identically distributed centred random variables, normalized so that $\text{Var}(\varphi_j(m)) = \frac{1}{M}$. Assume further that $\mathbb{E}(|\varphi(m)|^4) \leq \frac{B}{M^2}$, for some constant $B \geq 1$, and $N \geq C_0 M \log M$, for some constant $C_0$. Then, for each fixed $\alpha < 1 - \frac{1}{2c_0}$,
\[
S_{uFOS}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}},
\]
with probability at least \(1 - e^{-c_1M\log M}\), where constants \(c, c_1 > 0\) depend only on \(B, \alpha, \) and \(C_0\).

**Proof.** Fix \(\delta > 0\) and, for each \(v \in S^{M-1} \subset \mathbb{C}^M\), set

\[
G_\delta(v) = \{\varphi \in \mathbb{C}^M, \text{s.t. } |\langle v, \varphi \rangle| \geq 3\delta \text{ and } ||\varphi||_2 \leq 2\}.
\]

Let \(N_\delta \subset S^{M-1}\) be a \(\delta\)-net of the complex unit sphere \(S^{M-1}\). Then, for every \(x \in \mathbb{C}^M\) with \(||x||_2 = 1\), there exists \(v_x \in N_\delta\) with \(||x - v_x||_2 \leq \delta\). This defines a mapping \(x \mapsto v_x\). For each \(v \in N_\delta\), we consider the event

\[
E_v = \{|\Phi \cap G_\delta(v)| < \alpha N\}.
\]

**Claim 1:** \(\{S_{\text{uFOS}}(\Phi, \alpha N) < \delta\} \subseteq \bigcup_{v \in N_\delta} E_v\).

**Proof.** Let us take \(\omega \in (\bigcup_{v \in N_\delta} E_v)^c\). Denote by \(I_x \subset \{1, \ldots, N\}\) the set of indices corresponding to the vectors of \(\Phi\) which lie in \(G_\delta(v_x)\), that is,

\[
\Phi \cap G_\delta(v_x) = \{\varphi_j\}_{j \in I_x} = \Phi_{I_x}.
\]

Since \(\omega \in \bigcap_{v \in N_\delta} E_v^c\), we have that \(|I_x| \geq \alpha N\) for every \(x\). Moreover, for every \(i \in I_x\),

\[
|\langle x, \varphi_i \rangle| \geq |(v_x, \varphi_i)| - |(x - v_x, \varphi_i)| \geq |(v_x, \varphi_i)| - ||x - v_x||_2 ||\varphi_i||_2 \geq 3\delta - 2||x - v_x||_2 \geq \delta.
\]

Applying this inequality, we obtain

\[
S_{\text{uFOS}}(\Phi, \alpha N) = \min_{x \in \mathbb{C}^M} \max_{I \subset \{1, \ldots, N\}} \min_{i \in I} \min_{\|x\| = 1} |\langle x, \varphi_i \rangle| \geq \min_{x \in \mathbb{C}^M} \min_{\|x\| = 1} |\langle x, \varphi_i \rangle| \geq \delta.
\]

Thus, \(\omega \in \{S_{\text{uFOS}}(\Phi, \alpha N) < \delta\}^c\), and the claim is proven.

Then the union bound gives us the following estimate.

\[
P\{S_{\text{uFOS}}(\Phi, \alpha N) < \delta\} \leq \sum_{v \in N_\delta} P\{E_v\} \leq |N_\delta| \cdot \max_{v \in N_\delta} P\{E_v\}. \tag{A.1}
\]

Thus, to bound the probability \(P\{S_{\text{uFOS}}(\Phi, \alpha N) < \delta\}\), it is enough to bound \(|N_\delta|\) and \(\max_{v \in N_\delta} P\{E_v\}\).

**Claim 2**

There exists a \(\delta\)-net \(N_\delta\) of \(S^{M-1}\), such that \(|N_\delta| \leq (\frac{3}{\delta} + 1)^{2M}\).

**Proof.** Let us fix an embedding \(I\) of the unit sphere \(S^{M-1}\) into \(\mathbb{R}^{2M}\), given by \(I(x_0, \ldots, x_{M-1}) = (\mathcal{R}(x_0), \mathcal{S}(x_0), \ldots, \mathcal{R}(x_{M-1}), \mathcal{S}(x_{M-1}))\). Consider a dense packing of the real unit sphere \(I(S^{M-1}) = S_\mathbb{R}^{2M-1} \subset \mathbb{R}^{2M}\) with balls of radius \(\frac{\delta}{2}\). The distance between centers of two touching balls in the packing is \(\delta\), thus set of centers of these balls \(M_\delta\) form a \(\delta\)-net of \(S_\mathbb{R}^{2M-1}\) and \(N_\delta = \{I^{-1}(y)\}_{y \in M_\delta}\) is a \(\delta\)-net of \(S^{M-1}\).

To bound the cardinality of the constructed \(\delta\)-net \(N_\delta\), for each \(v \in N_\delta\) consider the open ball \(B_{\frac{\delta}{2}}(v) = \{x \in \mathbb{R}^{2M}, \text{s.t. } ||x - I(v)||_2 < \frac{\delta}{2}\}\) of radius \(\delta/2\). By

\(^1\)Claim 2 is a known fact, proof of which can be also found, for example, in [1].
construction of $\mathcal{N}_\delta$, these balls are disjoint and their union is contained in the ball $B_{\frac{\delta}{2}+1}(0) = \{ x \in \mathbb{R}^{2M}, \text{ s.t. } ||x||_2 < \frac{\delta}{2} + 1 \} \subset \mathbb{R}^{2M}$

$$\bigcup_{v \in \mathcal{N}_\delta} B_{\frac{\delta}{2}}(v) \subseteq B_{\frac{\delta}{2}+1}(0).$$

The volume of a ball $B(r)$ of radius $r$ in $\mathbb{R}^{2M}$ is given by $\text{Vol}(r) = \frac{\pi^M}{M!} r^{2M}$. Then the volume comparison gives

$$\frac{\pi^M}{M!} |\mathcal{N}_\delta| \left( \frac{\delta}{2} \right)^{2M} = \text{Vol} \left( \bigcup_{v \in \mathcal{N}_\delta} B_{\frac{\delta}{2}}(v) \right) \leq \text{Vol} \left( B_{\frac{\delta}{2}+1}(0) \right) = \frac{\pi^M}{M!} \left( 1 + \frac{\delta}{2} \right)^{2M}. $$

This implies $|\mathcal{N}_\delta| \leq \left( 1 + \frac{\delta}{2} \right)^{2M}$, as claimed. \(\square\)

To estimate $\mathbb{P}\{\mathcal{E}_v\}$, note that

$$\mathbb{P}\{\mathcal{E}_v\} = \mathbb{P}\left\{ \sum_{i=1}^{N} 1_{\{\varphi_i \notin G_\delta(v)\}} \geq (1 - \alpha) N \right\},$$

that is, the probability $\mathbb{P}\{\mathcal{E}_v\}$ equals to the tail probability of the sum of $N$ independent Bernoulli random variables $1_{\{\varphi_i \notin G_\delta(v)\}}$, $i \in \{1, \ldots, N\}$, with success probability $p$ to be estimated. To bound this tail probability, we are going to apply Hoeffding’s inequality, namely,

$$\mathbb{P}\left( \sum_{i=1}^{N} 1_{\{\varphi_i \notin G_\delta(v)\}} \geq N(p + t) \right) \leq e^{-2Nt^2}, \quad (A.2)$$

for every $t > 0$. We bound the success probability $p$ of the Bernoulli random variable $1_{\{\varphi_i \notin G_\delta(v)\}}$ by

$$p = \mathbb{P}\{\varphi_i \notin G_\delta(v)\} \leq \mathbb{P}\{|\langle v, \varphi_i \rangle| < 3\delta\} + \mathbb{P}\{|||\varphi_i|| > 2\}.$$

To bound $\mathbb{P}\{|\langle v, \varphi_i \rangle| < 3\delta\}$, we use the small ball probability estimates, and the bound for $\mathbb{P}\{|||\varphi_i|| > 2\}$ follows from the assumptions on the distribution of random variables $\varphi_i$, $i \in \{1, \ldots, N\}$.

**Claim 3:** Let $v \in \mathcal{N}_\delta$ be such that its spread part $\hat{v}$ is well-defined for some $0 < K_1 < K_2$. Then, for any $\varepsilon \in (0, 1/2)$,

$$\mathbb{P}\{|\langle v, \varphi_i \rangle| < 3\delta\} \leq \tilde{C} \left( 3\delta \sqrt{M} + \frac{K_2}{(1 - \varepsilon) \sqrt{M}} \right) + Ce^{-c\varepsilon^2 M},$$

where constants $\tilde{C}, C, c$ depend only on $K_1, K_2$ and $B$.

**Proof.** Let us renormalize random variables $\varphi_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, by setting $\tilde{\varphi}_j(m) = \sqrt{M} \varphi_j(m)$. Then new variables $\tilde{\varphi}_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, satisfy the assumptions of the Theorem A.1.3. Using this result, we obtain the following inequality for any $\varepsilon, \beta \in (0, 1/2)$

$$\mathbb{P}\{|\langle v, \varphi_i \rangle| < 3\delta\} = \mathbb{P}\{|\langle v, \tilde{\varphi}_i \rangle| < 3\delta \sqrt{M}\} \leq \frac{C}{\sqrt{M}} \left( 3\delta \sqrt{M} + \frac{1}{D_{\varepsilon, \beta M}(\hat{v}) \sqrt{M}} \right) + Ce^{-c\varepsilon^2 \beta M},$$
where $\hat{v}$ is the spread part of $v$, defined for some $0 < K_1 < K_2$.

Let us prove that, for any $v \in \mathcal{N}_\delta$ and any $\varepsilon, \beta \in (0, 1/2)$, $D_{\varepsilon, \beta M}(\hat{v}) \geq \frac{1-\varepsilon}{K_2}$. This would allow us to conclude our claim. By the definition of the essential LCD, $D_{\varepsilon, \beta M}(\hat{v})$ is the infimum over all $t > 0$, such that all except $\beta M$ entries of $t\hat{v}$ are $\varepsilon$-close to non-zero integers. That is, the infimum is taken over all $t > 0$, such that for some integers $m_i \neq 0$,

$$|t|\hat{v} - |m_i| \leq |t\hat{v} - m_i| \leq \varepsilon.$$ 

Consequently, $|m_i| - \varepsilon \leq t|\hat{v}| \leq |m_i| + \varepsilon$,

$$t \geq \frac{|m_i| - \varepsilon}{|\hat{v}|} \geq \frac{1-\varepsilon}{K_2},$$

since $|\hat{v}| \leq K_2$ for each $i \in \sigma(v)$ by definition (see also Example A.1.2, part (2)). Thus, $D_{\varepsilon, \beta M}(\hat{v}) \geq \frac{1-\varepsilon}{K_2}$, which completes the proof. 

**Claim 4:** $P\{||\varphi_i||^2 > 2\} \leq \frac{B-1}{9M}$, for each $i \in \{1, \ldots, N\}$.

**Proof.** For each $i \in \{1, \ldots, N\}$, we introduce a new random variable $\bar{X}_i$ given by $\bar{X}_i = \sum_{m \in \mathbb{Z}_M} |\varphi_i(m)|^2$. Then

$$E(\bar{X}_i) = \sum_{m \in \mathbb{Z}_M} E(|\varphi_i(m)|^2) = \sum_{m \in \mathbb{Z}_M} \text{Var}(\varphi_i(m)) = M \frac{1}{M} = 1,$$

$$\text{Var}(\bar{X}_i) = E\left( \sum_{m \in \mathbb{Z}_M} |\varphi_i(m)|^2 \right)^2 - (E(\bar{X}_i))^2$$

$$= \sum_{m \in \mathbb{Z}_M} E(|\varphi_i(m)|^4) + \sum_{m, m' \in \mathbb{Z}_M, m \neq m'} \text{Var}(\varphi_i(m)) \text{Var}(\varphi_i(m')) - 1$$

$$\leq M \frac{B}{M^2} + \frac{M(M-1)}{M^2} - 1 = \frac{B-1}{M}.$$ 

Then by Chebyshev’s inequality we obtain

$$P\{|\bar{X}_i - E(\bar{X}_i)| \geq 3\} \leq \frac{\text{Var}(\bar{X}_i)}{9}.$$ 

Thus

$$P\{||\varphi_i||^2 > 4\} \leq \frac{B-1}{9M},$$

which completes the proof. 

Claims 3 and 4 allows us to conclude that

$$p = P\{\varphi \notin G_\delta(v)\} \leq P\{||v, \varphi_i|| < 3\delta\} + P\{||\varphi_i|| > 2\}$$

$$\leq \tilde{C} \left( 3\delta \sqrt{M} + \frac{K_2}{(1-\varepsilon)\sqrt{M}} \right) + C e^{-\alpha^2 M} + \frac{B-1}{M} = \tilde{p}.$$ 

Assuming, $M$ is sufficiently large, and $\delta \leq \frac{\varepsilon}{\sqrt{M}}$ for $c$ sufficiently small, we obtain $\tilde{p}$ close to zero, which makes this estimate meaningful. Note that, replacing $p$ in the
left-hand side of the inequality (A.2) with some $\hat{p} \geq p$ will not increase the resulting probability. As such, by applying the inequality (A.2) with $t = 1 - \alpha - \hat{p}$, we obtain

$$\mathbb{P}\{\mathcal{E}_r\} = \mathbb{P}\left\{\sum_{i=1}^{N} 1_{\{\varphi_i \notin \mathcal{G}_s(v)\}} \geq (1 - \alpha)N\right\} \leq e^{-2N(1-\alpha-\hat{p})^2}$$

$$= \exp\left(-2N\left((1 - \alpha) - \tilde{C}\left(3\delta \sqrt{M} + \frac{K_2}{(1 - \varepsilon)\sqrt{M}}\right) - Ce^{-\varepsilon^2M} - \frac{B - 1}{9M}\right)^2\right).$$

Now, we substitute bounds for $\mathbb{P}\{\mathcal{E}_r\}$ and $\mathcal{N}_\delta$ in inequality (A.1) and take the logarithm to obtain

$$\log(\mathbb{P}\{\mathcal{S}_{uFOS}(\Phi, \alpha N) < \delta\}) \leq \log(\mathbb{P}\{\mathcal{E}_r\}) + \log(|\mathcal{N}_\delta|) \leq -2N(1 - \alpha - \hat{p})^2 + 2M \log\left(\frac{2}{\varepsilon} + 1\right).$$

Choose $\delta = \frac{c}{\sqrt{M}}$ with $c$ small enough, so that $\hat{p}$ is sufficiently close to zero. Since $N \geq C_0 \log M$, we have

$$\log(\mathbb{P}\{\mathcal{S}_{uFOS}(\Phi, \alpha N) < \delta\}) \leq -2C_0 M \log M (1 - \alpha - \hat{p})^2 + 2M \log\left(\frac{3\sqrt{M}}{c}\right)$$

$$= -2C_0 M \log M \left((1 - \alpha - \hat{p})^2 - \frac{1}{2C_0} - \frac{\log(3/c)}{C_0 \log M}\right).$$

To conclude the proof, we need to show that $(1 - \alpha - \hat{p})^2 - \frac{1}{2C_0} - \frac{\log(3/c)}{C_0 \log M} > 0$. By choosing $M$ sufficiently large, we can make $\frac{\log(3/c)}{C_0 \log M}$ close to zero, so that $\frac{\log(3/c)}{C_0 \log M} \leq \varepsilon << 1$. Then, if we choose $\alpha < 1 - \hat{p} - \frac{1}{\sqrt{2C_0}} - \sqrt{\varepsilon}$, we obtain

$$\mathcal{S}_{uFOS}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}}$$

with probability at least $1 - e^{-c_1 M \log M}$, for some constant $c_1 > 0$. \hfill \Box

### A.3 Large frame coefficients

In this section, we obtain a bound on the $\beta$-largest uniform frame order statistics for a random frame with independent frame vectors under the fourth moment assumption. The proof of Theorem A.3.1 follows the same main steps as the proof of Theorem A.2.1.

**Theorem A.3.1.** Let the frame $\Phi = \{\varphi_j\}_{j=1}^{N} \subset \mathbb{C}^M$, with $M$ large enough, be such that $\varphi_j(m)$, $j \in \{1, \ldots, N\}$, $m \in \mathbb{Z}_M$, are independent identically distributed centred random variables, normalized so that $\text{Var}(\varphi_j(m)) = \frac{1}{M}$. Assume further that $\mathbb{E}(|\varphi_j(m)|^4) \leq \frac{B}{M^4}$, for some constant $B \geq 1$, and $N \geq C_0 \log M$, for some constant $C_0$. Then, for each fixed $\beta < 1 - \frac{1}{2C_0}$,

$$\mathcal{L}_{uFOS}(\Phi, \beta N) \leq \frac{K}{\sqrt{M}}$$

with probability at least $1 - e^{-c_1 M \log M}$, where constants $K, c_1 > 0$ depend only on $B$, $\beta$, and $C_0$. 

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Proof. Fix $\delta > 0$ and, for each $x \in S^{M-1} \subset \mathbb{C}^M$, set
\[
F_\delta(x) = \left\{ \varphi \in \mathbb{C}^M, \text{ s.t. } |\langle x, \varphi \rangle| \leq \frac{\delta}{3} \text{ and } ||\varphi||_2 \leq 2 \right\}.
\]

Let $\mathcal{N}_3^\delta \subset S^{M-1}$ be a $\frac{\delta}{3}$-net of the complex unit sphere $S^{M-1}$. Then for every $x \in S^{M-1}$, there exists $v_x \in \mathcal{N}_3^\delta$ such that $||x - v_x||_2 \leq \frac{\delta}{7}$. For each $v \in \mathcal{N}_3^\delta$, we define
\[
\mathcal{W}_v = \{ |\Phi \cap F_\delta(v)| < \beta N \}.
\]

Claim 1: $\{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \delta \} \subseteq \bigcup_{v \in \mathcal{N}_3^\delta} \mathcal{W}_v$.

Proof. Let us take $\omega \in \left( \bigcup_{v \in \mathcal{N}_3^\delta} \mathcal{W}_v \right)^c$. For each $x \in S^{M-1}$, let us denote by $\mathcal{J}_x \subset \{1, \ldots, N\}$ the set of indices, such that $\Phi \cap F_\delta(v_x) = \{ j \}_{j \in \mathcal{J}_x} = \Phi_{\mathcal{J}_x}$. Since $\omega \in \bigcap_{v \in \mathcal{N}_3^\delta} \mathcal{W}_v^c$, we have that $|\mathcal{J}_x| \geq \beta N$ for every $x$. Moreover, for every $i \in \mathcal{J}_x$,
\[
|\langle x, \varphi_i \rangle| \leq |\langle v_x, \varphi_i \rangle| + |\langle x - v_x, \varphi_i \rangle| \leq |\langle v_x, \varphi_i \rangle| + ||x - v_x||_2 ||\varphi_i||_2 \leq \frac{\delta}{3} + 2 ||x - v_x||_2 \leq \delta.
\]

Applying this inequality, we obtain
\[
\mathcal{L}_{uFOS}(\Phi, \beta N) = \max_{x \in \mathbb{C}^M, |\mathcal{J}\subset\{1, \ldots, N\}, |\mathcal{J}| \geq \beta N} \min_{|x||=1} \max_{i \in \mathcal{J}} |\langle x, \varphi_i \rangle| \leq \max_{x \in \mathbb{C}^M, |i||=1} \max_{|\mathcal{J}| \geq \beta N} |\langle x, \varphi_i \rangle| \leq \delta.
\]

Thus, $\omega \in \{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \delta \}^c$, and the claim is proven.

Then the union bound gives us the estimate
\[
\mathbb{P}\{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \delta \} \leq \sum_{v \in \mathcal{N}_3^\delta} \mathbb{P}\{ \mathcal{W}_v \} \leq \mathcal{N}_3^\delta \max_{v \in \mathcal{N}_3^\delta} \mathbb{P}\{ \mathcal{W}_v \}.
\]

That is, to bound the probability $\mathbb{P}\{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \delta \}$, it is enough to bound $|\mathcal{N}_3^\delta|$ and $\max_{v \in \mathcal{N}_3^\delta} \mathbb{P}\{ \mathcal{W}_v \}$.

As we showed before (see Claim 2 in the proof of Theorem A.2.1), there exists a $\frac{\delta}{3}$-net $\mathcal{N}_3^\delta$ of $S^{M-1}$, such that
\[
|\mathcal{N}_3^\delta| \leq \left( \frac{6}{\delta} + 1 \right)^{2M}.
\]

To estimate $\mathbb{P}\{ \mathcal{W}_v \}$, we note that
\[
\mathbb{P}\{ \mathcal{W}_v \} = \mathbb{P}\left\{ \sum_{i=1}^N 1_{\{ \varphi_i \notin F_\delta(v) \}} \geq (1 - \beta)N \right\},
\]

that is, the probability $\mathbb{P}\{ \mathcal{W}_v \}$ equals to the tail probability of the sum of $N$ independent Bernoulli random variables $1_{\{ \varphi_i \notin F_\delta(v) \}}$, $i \in \{1, \ldots, N\}$, with success probability
\[
p = \mathbb{P}\{ \varphi_i \notin F_\delta(v) \} \leq \mathbb{P}\{ |\langle v, \varphi_i \rangle| > \frac{\delta}{3} \} + \mathbb{P}\{ ||\varphi_i|| > 2 \}.
\]

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As we showed before (see Claim 4 in the proof of Theorem A.2.1), for each \( i \in \{1, \ldots, N\} \),
\[
\mathbb{P}\{|\|\varphi_i\|_2 > 2\| \leq \frac{B - 1}{9M}.
\]

Now it remains to estimate \( \mathbb{P}\{ |\langle v, \varphi_i \rangle | > \frac{2}{3} \} \), which we do in the following claim.

**Claim 2:** For any \( x \in \mathbb{S}^{M-1} \) and \( k > 0 \)
\[
\mathbb{P}\left\{ |\langle x, \varphi_i \rangle |^2 > \frac{k\sqrt{B+1} + 1}{M} \right\} \leq \frac{1}{k^2}.
\]

**Proof.** We start with estimating the expectation and variance of the random variable \( |\langle x, \varphi_i \rangle |^2 \).

\[
\mathbb{E}\left( |\langle x, \varphi_i \rangle |^2 \right) = \mathbb{E}\left( \sum_{m \in \mathbb{Z}_M} |x(m)|^2 |\varphi_i(m)|^2 + \sum_{m_1, m_2 \in \mathbb{Z}_M, \ m_1 \neq m_2} x(m_1)x(m_2)\varphi_i(m_1)\overline{\varphi_i(m_2)} \right)
\]
\[
= \sum_{m \in \mathbb{Z}_M} |x(m)|^2 \text{Var}(\varphi_i(m)) = \frac{1}{M};
\]

\[
\text{Var}\left( |\langle x, \varphi_i \rangle |^2 \right) = \mathbb{E}\left( \sum_{m_1, m_2 \in \mathbb{Z}_M} x(m_1)x(m_2)\overline{x(m_3)x(m_4)\varphi_i(m_1)\overline{\varphi_i(m_2)}\varphi_i(m_3)\overline{\varphi_i(m_4)}} \right) - \mathbb{E}^2\left( |\langle x, \varphi_i \rangle |^2 \right)
\]
\[
= \mathbb{E}\left( \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}_M} |x(m_1)|^4 |\varphi_i(m_1)|^4 + 2\mathbb{E}\left( \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}_M} |x(m_1)|^2 |x(m_2)|^2 |\varphi_i(m_1)|^2 |\varphi_i(m_2)|^2 \right) \right)
\]
\[
+ 2\mathbb{E}\left( \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}_M} |x(m_1)|^2 |x(m_2)|^2 |\varphi_i(m_1)|^2 |\varphi_i(m_2)|^2 \right) - \frac{1}{M^2}
\]
\[
\leq \sum_{m \in \mathbb{Z}_M} |x(m)|^4 \mathbb{E}\left( |\varphi_i(m)|^4 \right) + 3 \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}_M, \ m_1 \neq m_2} |x(m_1)|^2 |x(m_2)|^2 \text{Var}(\varphi_i(m_1)) \text{Var}(\varphi_i(m_2)) - \frac{1}{M^2}
\]
\[
\leq \frac{B}{M^2} \sum_{m \in \mathbb{Z}_M} |x(m)|^4 + \frac{3}{M^2} \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}_M, \ m_1 \neq m_2} |x(m_1)|^2 |x(m_2)|^2 - \frac{1}{M^2} \leq \frac{B + 1}{M^2}.
\]

Here in the last step we use that \( |x|_2^2 = 1 \), thus \( \sum_{m \in \mathbb{Z}_M} |x(m)|^4 \leq 1 \) and

\[
\sum_{m_1, m_2 \in \mathbb{Z}_M, \ m_1 \neq m_2} |x(m_1)|^2 |x(m_2)|^2 \leq \sum_{m_1, m_2 \in \mathbb{Z}_M} |x(m_1)|^2 |x(m_2)|^2 = \left( \sum_{m \in \mathbb{Z}_M} |x(m)|^2 \right)^2 = 1.
\]

Using Chebyshev’s inequality we conclude that, for any \( k > 0 \),
\[
\mathbb{P}\left\{ |\langle x, \varphi_i \rangle |^2 > \frac{k\sqrt{B+1} + 1}{M} \right\} \leq \mathbb{P}\left\{ \left| \frac{|\langle x, \varphi_i \rangle |^2 - 1}{M} \right| \geq \frac{k\sqrt{B+1}}{M} \right\} \leq \frac{1}{k^2}.
\]
Let us fix some \( k > 0 \). Then, setting \( \delta = \frac{3\sqrt{k}\sqrt{B} + 1 + 1}{\sqrt{M}} \) and using Claim 2, we conclude that

\[
p = \mathbb{P}\{\phi_i \notin F_\delta(v)\} \leq \mathbb{P}\left\{ |\langle v, \phi_i \rangle| > \frac{\delta}{3} \right\} + \mathbb{P}\{||\phi_i|| > 2\} \leq \frac{1}{k^2} + \frac{B - 1}{M} = \tilde{p}.
\]

Assuming \( M \) and \( k \) are sufficiently large, we obtain \( \tilde{p} \) is close to zero, which makes this estimate meaningful. Applying Hoeffding’s inequality (Lemma 2.2.5) with \( t = 1 - \beta - \tilde{p} \), we obtain

\[
\mathbb{P}\{W_v\} = \mathbb{P}\left\{ \sum_{i=1}^{N} 1_{\{\phi_i \notin F_\delta(v)\}} \geq (1 - \beta)N \right\} \leq e^{-2N(1-\alpha-\tilde{p})^2}
\]

\[
= \exp\left(-2N \left(1 - \alpha - \frac{1}{k^2} - \frac{B - 1}{9M}\right)^2\right).
\]

Now, we substitute bounds for \( \mathbb{P}\{W_v\} \) and \( |N_\frac{\epsilon}{2}| \) into the inequality (A.3) and take the logarithm to obtain

\[
\log\left(\mathbb{P}\left\{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \frac{3C(k, B)}{\sqrt{M}} \right\}\right) \leq \log(\mathbb{P}\{W_v\}) + \log\left(|N_\frac{\epsilon}{2}|\right)
\]

\[
\leq -2N(1 - \beta - \tilde{p})^2 + 2M \log\left(\frac{2\sqrt{M}}{C(k, B)} + 1\right),
\]

where \( C(k, B) = \sqrt{k\sqrt{B} + 1 + 1} \). Since \( N \geq C_0 M \log M \), we have

\[
\log\left(\mathbb{P}\left\{ \mathcal{L}_{uFOS}(\Phi, \beta N) > \frac{3C(k, B)}{\sqrt{M}} \right\}\right) \leq -2C_0 M \log M(1 - \alpha - \tilde{p})^2 + 2M \log\left(\frac{3\sqrt{M}}{C(k, B)}\right)
\]

\[
= -2C_0 M \log M \left((1 - \alpha - \tilde{p})^2 - \frac{1}{2C_0} - \frac{\log(3/C(k, B))}{C_0 \log M}\right).
\]

To conclude the proof, we need to show that \( (1 - \alpha - \tilde{p})^2 - \frac{1}{2C_0} - \frac{\log(3/C(k, B))}{C_0 \log M} > 0 \). By choosing \( M \) sufficiently large, we can make \( \frac{\log(3/C(k, B))}{C_0 \log M} \) arbitrarily close to zero, so that \( \frac{\log(3/C(k, B))}{C_0 \log M} \leq \epsilon << 0 \). Then, if we choose \( \alpha < 1 - \tilde{p} - \frac{1}{\sqrt{2C_0}} - \sqrt{\epsilon} \), we obtain

\[
\mathcal{L}_{uFOS}(\Phi, \beta N) > \frac{3C(k, B)}{\sqrt{M}}
\]

with probability at least \( 1 - e^{-c_1 M \log M} \), for some constant \( c_1 > 0 \). \( \square \)
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